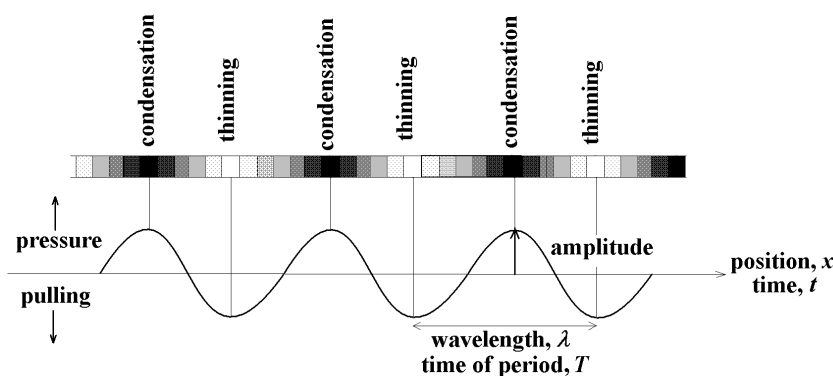


Mechanical waves, acoustics and ultrasounds.

Lecturer: Péter Maróti

The energy of an oscillating particle in elastic medium can propagate both in space and in time. The phenomenon is called *mechanical wave*. It is important to stress that not the particle itself but its state of motion (e.g. energy or momentum) is spread out. In general, one can talk about *propagating wave* if a disturbance in the medium extends in its surroundings 1) along a straight line (one dimensional wave), 2) along a surface (two dimensional waves, e.g. the water surface waves) and 3) in the space (three dimensional wave e.g. sound, electromagnetic (light, radio and micro) etc. waves).

Propagation of the wave along straight line; transverse and longitudinal waves. The waves can be classified according to the direction of motion of particles of the elastic medium relative to the direction of the propagation of the wave. If the particles are moving in the plane perpendicular to the direction of the propagation of the wave, the wave is called *transverse (transversal) wave*. If the direction of the movement of the oscillating particles falls in the direction of the propagation of the wave, we are talking about *longitudinal wave*. If a snap shot (picture) is taken from the wave (t is constant), peaks/valleys and condensations/thinnings can be identified in the transversal and longitudinal waves,



respectively. The particles are oscillating mainly around their equilibrium positions therefore their average displacement is zero. Not the particle but its state of motion (energy, phase, momentum etc.) propagates in the elastic medium in form of wave. As sharing forces can occur in solid state of the

medium, both transverse and longitudinal waves may arise in this medium. In contrast to solid states, there are no sharing forces in fluids and gases, consequently longitudinal (and no transverse) waves can occur in these media.

Equation of the one dimensional harmonic wave. The kinematic relationship of the harmonic oscillation (position y vs. time t) is

$$y(x=0, t) = A \sin(\omega t + \varphi).$$

This is the displacement of the source of the wave at an arbitrary time t . That state of the vibration has not reached the particle at distance x from the source but the particle preserves an earlier state of vibration that belongs to an earlier time ($t - x/c$). Here c denotes the *velocity of propagation of the wave*. At that moment, the displacement of the source of the wave was $A \sin(\omega(t - x/c) + \varphi)$. Therefore, the displacement of the particle at position x and at time t can be given by the following expression:

$$y(x, t) = A \sin(\omega(t - x/c) + \varphi).$$

For undamped and simple harmonic (sinusoidal) wave, the vibration state of a particle in the elastic medium is determined by the *phase* (argumentum) of the sine function, which is $(\omega(t - x/c) + \varphi)$.

The smallest distance between displacements of equal phase is called *wavelength* (λ), that can be expressed by the speed of propagation (c), the *time of period* (T), the *linear frequency* (f) and the *angular frequency* (ω) in the following way:

$$\lambda = c \cdot T = \frac{c}{f} = 2\pi \frac{c}{\omega}.$$

The equation of the one dimensional wave is

$$y(x,t) = A \sin 2\pi \left(\frac{t}{T} - \frac{x}{\lambda} + \varphi / 2\pi \right).$$

One can see that the wave propagates periodically in time and space. The temporal periodicity is characterized by the time of period (T) and the spatial periodicity is described by the wavelength (λ). Similar quantity-pair is the angular frequency $\omega = 2\pi/T$ (the number of vibrations during 2π seconds) and the *wave number* $\bar{\lambda} = \frac{2\pi}{\lambda}$ (the number of waves in propagation path of 2π centimeters):

$$y(x,t) = A \sin(\omega \cdot t - \bar{\lambda} \cdot x + \varphi).$$

Reflection and refraction of (acoustic) wave (sound). If the speeds of propagation of the wave in two media are different, then the wave arriving at the interface of the media will be partly reflected (R) and partly transmitted and refracted ($T = 1 - R$). Because the frequencies of the wave in the two media should be equal: $f_1 = f_2$, the ratio of the speeds and the wavelengths in the media are also equal: $\frac{c_1}{c_2} = \frac{\lambda_1}{\lambda_2}$. If the wave speeds up, the wavelength increases; if the wave slows down, the wavelength decreases.

One dimensional wave. Standing wave. A wave of amplitude A_0 travels along a stretched string of mass density μ_1 . If this string is joined to a second string of mass density μ_2 , part of the wave is reflected with amplitude A_R at the boundary, and part is transmitted with amplitude A_T :

$$A_R = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} A_0 \quad A_T = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} A_0.$$

Note, if the wave travels from a thin string into a heavy string ($\mu_1 < \mu_2$), the reflected wave will be inverted (shown in these equations as a negative amplitude). If the wave is reflected from a *fixed* end (or from a boundary where the inertia of the medium increases), the reflected wave will experience 180° phase change (*i.e.*, it will be inverted). If the wave is reflected from a *free* end (or from a boundary where the inertia of the medium decreases), the reflected wave will be in phase with the incident wave (*i.e.*, it will be erected). In either case, the transmitted wave will be in phase with the incident wave.

When identical waves traveling in opposite directions are superimposed, *standing waves* occur. The distance between adjacent *nodes* in a standing wave pattern is *half* the wavelength, but the amplitude of oscillation at the *antinodes* is *twice* the amplitude of each of the component waves.

Standing waves occur in pipes: the fundamental and overtone (harmonics) frequencies are the same in pipes that are open at both ends and pipes that are closed at both ends. Pipes open at one end and closed at the other differ in two ways from symmetric pipes. 1) The fundamental frequency in an open/closed pipe is *half* the fundamental frequency of a symmetric (closed/closed or open/open) pipe of the same length. 2) The open/closed pipe produces only the odd-numbered multiples of the fundamental frequency, while the symmetric pipes produce *all* integer multiples of the fundamental.

Two- and three dimensional waves. The directions of the reflected and the refracted waves are in the same plane. The angles of incidence and reflection are equal. The ratio of the sine of the angle of incidence (α) and the angle of refraction (β) is equal to the ratio of the speeds of the wave in the two media (*Snell's law of refraction*):

$$\frac{\sin \alpha}{\sin \beta} = \frac{c_1}{c_2} = n_{21} = \text{const.},$$

where n_{21} is the *refractive index* of the medium 2 relative to that of medium 1. If $c_1 > c_2$, then medium 1 has smaller acoustic density (it is more loose but not in mechanical sense) than medium 2. If the wave comes from a medium of less acoustic density to a medium of larger acoustic density, then $\alpha > \beta$ and the refractive index will be larger than 1: $n_{21} > 1$. The relationships are symmetric: waves propagating in opposite direction show $\alpha < \beta$ and $n_{12} > 1$.

	air	water	skin
c (m/s)	345	1480	1950
acoustic density	large	small	very small
critical angle, α_{crit} (degree)	13.5		49.4

Under this condition, the wave in medium 2 will not penetrate into medium 1 and will be reflected from the boundary if the angle of incidence becomes

greater than a critical value: $\sin \alpha_{\text{crit}} = c_1/c_2$. This is the *critical angle of total reflection* ($\beta = 90^\circ$). Above the critical angle, the Snell's law cannot be applied and becomes meaningless. According to the experiences, the intensity of the reflected wave gradually increases and that of the refracted wave decreases upon increase of the angle of incidence from 0 to the critical value. At the critical angle of incidence, the refracted wave disappears and the reflected wave remains only with the same intensity as that of the incident wave (this is why the reflection is called „total“). The total reflection may play tricks on the echographers because, in lack of careful planning, the ultrasound will not penetrate into tissues under investigation.

Special case: the reflected fraction of the wave energy hitting perpendicularly the boundary of two media is

$$R = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1} \right)^2,$$

and that of the penetrated (transmitted) wave is $T = 1 - R$. Here $Z = \rho \cdot c$ is called the *acoustic impedance (resistance)* of the medium, where ρ is the mechanical density and c is the speed of the wave. Reflection can occur at the boundary of two media of different acoustic impedances only: $R \neq 0$, if $Z_1 \neq Z_2$.

Example. Ultrasound is directed from air ($Z = 0.43 \cdot 10^3 \text{ kg} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$) perpendicularly to soft tissues ($Z = 1.6 \cdot 10^6 \text{ kg} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$). The reflected portion is $R = 0.9994$, *i.e.* the transmission is $T = 0.06\%$ only. If, however, water-base cellulose jelly as *acoustic coupling agent* ($Z = 1.5 \cdot 10^6 \text{ kg} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$) is used between the transducer and the soft tissue, then the reflected fraction diminishes to $R = 0.001$, *i.e.* the vast majority of the wave ($T = 0.999$) invades the tissue. The loss is at least 3 orders of magnitude if no appropriate acoustic coupling is applied.

Energy of the harmonic mechanical waves. The energy of the wave is the energy content of the section of the medium where the wave propagates. The total energy consists of kinetic and elastic terms. As the volume section ΔV of the medium has a mass of $\Delta m = \rho \cdot \Delta V$ which carries out harmonic vibration of angular frequency ω and amplitude A , the total energy is $E_{\text{total}} = \frac{1}{2} \Delta m A^2 \omega^2$. The time average of the *energy density* (energy of the unit volume) is

$$\bar{w} = \frac{\langle E_{\text{total}} \rangle}{\Delta V} = \frac{1/2 \cdot \Delta m A^2 \omega^2}{\Delta V} = \frac{1}{2} \rho A^2 \omega^2,$$

i.e. linearly proportional to the density of the medium and to the square of the amplitudes and angular frequencies.

Radiation power. In extended elastic medium, a plane wave of energy density \bar{w} passes perpendicularly a surface element of area q at $t = 0$. After 1 s, the wave front will be at a distance c from that position. During that time, the wave fills a prism of volume $q \cdot c$ with energy:

$$P = \bar{w} q c.$$

If the wave is not plane wave, the surface should be divided into small (infinitesimal) sections to be able to consider the wave front as plane. The total radiation power will be the sum (integral) of the infinitesimal radiation powers obtained for the small surface sections.

Intensity (power density). The intensity of the wave is the energy passing through unit surface area perpendicular to the direction of propagation within unit time. It can be expressed in different equivalent forms:

$$I = \frac{P}{q} = \bar{w} c = \frac{1}{2} \rho c A^2 \omega^2 = \frac{1}{2} \rho c v_{\text{max}}^2 = \frac{1}{2} \frac{p_{\text{max}}^2}{\rho c},$$

the dimension is power/area and the unit is W/m².

Example. Ultrasound of intensity 100 mW/cm², amplitude 2 nm and frequency 3 MHz is traveling in water (density 10³ kg/m³ and speed 1480 m/s). The maxima of the speed, acceleration and pressure are $v_{\text{max}} = 3.7$ cm/s (small), $a_{\text{max}} = 7 \cdot 10^4$ g (extremely large!) and $p_{\text{max}} = 0.5$ bar (moderately large), respectively.

Distance-dependence of the intensity. The fronts of the waves emitted by point-like source in homogeneous and isotropic medium are concentric spheres. The energy emitted by the source within unit time, P is equal to the total energy transmitted through the spherical surface $q = 4\pi r^2$. The intensity

$$I = \frac{P}{q} = \frac{P}{4\pi r^2}$$

changes as inverse square of the distance from the point-like source r .

The distance- and direction-dependence of the intensity of an ultrasound source (transducer) of frequency f and diameter D is much more complex. Near- and far-field zones can be distinguished in the spatial distribution of the ultrasound. The boundary of the two fields located at distance N from the transducer (*focal length*) is often called as *focal plane*:

$$N = f \cdot D^2 / (4c).$$

In the near field zone, the ultrasound is well bundled and consists of parallel waves (beams). In the far-field zone, however, the wave is divergent. The degree of divergence (characterized by the angle α) can be derived from diffraction theory applied to a diaphragm (slit) of diameter D (consult the lecture on optics):

$$\sin \alpha = 1,22 \cdot c / (fD).$$

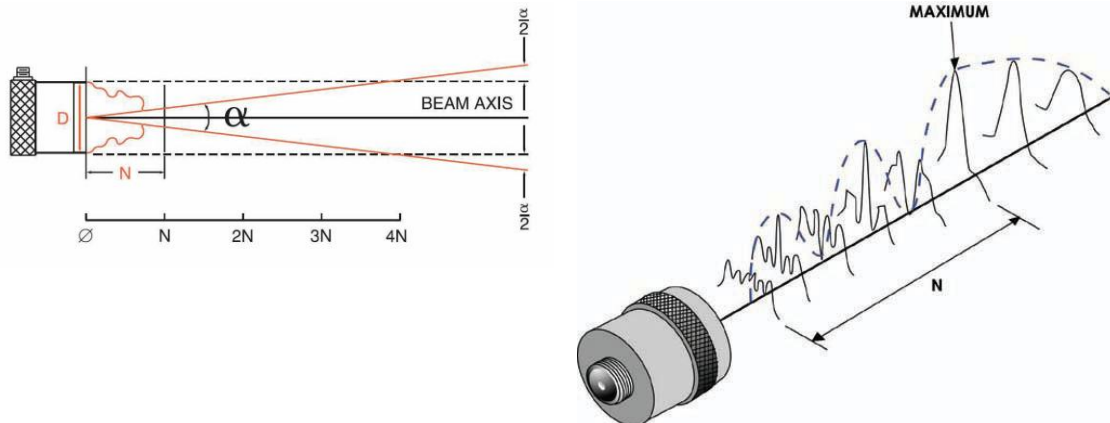
f (MHz)	N (cm)	α (degree)
1	1.6	12.3
2	3.2	6.1
5	7.9	2.5

Example. The speed of ultrasound in soft tissue is $c = 1580$ m/s. The diameter of the transducer is $D = 1$ cm! The calculated and coupled values are included in the table.

From the point of view of the medical diagnostics, the use of the near-field is recommended because of the small divergence of the ultrasound in

this zone. In the far-field zone, the drawbacks are the divergence of the beams and the appearance of possible side lobes which could be the source of different artifacts in the image.

The spatial variation of the intensity along the axis of propagation provides complex pattern, as well. There are frequently changing minima and maxima in the near-field zone. The spatial variation of the intensity is significantly smoothed in the far-field zone and the intensity will drop monotonously with increasing distance from the transducer. In direction perpendicular to the propagation, the spatial change of the intensity shows complex pattern (see the figure).



The divergence of the wave can be reduced by proper *focusing* with acoustic and/or electric lenses. The divergent waves can be collected by acoustic lenses (similarly as the convex lens collects the parallel light beams into the focal point). The method of electric focusing is based on time-dependent triggering of the elements of the array in the transducer. The phases of the emitted elementary waves are governed by proper timing (*phase-control*) and the resultant wave front can be made convergent (see the *Huygens-Fresnel's principle* in the optics).

Objective sound intensity. In the table below, the sound power of some sound sources are listed. The values are very different and cover wide range. To facilitate more convenient

Source of sound	P (W)
normal talk	10^{-5}
shout	10^{-3}
piano (maximum)	0,1
horn (car)	5
laud-speaker	10^2
horn for anti-aircraft defense	10^3

comparison among powers (P_1 and P_2) or intensities (I_1 and I_2) of sound sources, logarithmic scale is introduced. The difference of levels of powers (or intensities) of two sources is given by

$$n = 10 \cdot \lg \frac{P_2}{P_1} \quad \text{decibel (dB)}.$$

Accordingly, the power of a megaphone is 50 dB (5 orders of magnitude) larger than that of the shout.

Subjective sound intensity, audibility, loudness. The objective sound intensity is measured by physical instruments (e.g. by microphones). It differs from the subjective intensity that the human ear senses. The subjective intensity is not linearly proportional to the intensity of the sound: the sounds of 100 times and 1000 times larger intensities evoke not 100 times and 1000 times stronger sensation in the ear but the amplifications are 20 and 30 dB only. The objective and subjective intensities resemble of the stimulus and feeling of the perception, respectively. The *Weber-Fechner law* of psychophysics connects the stimulation with the sensation in the form of a logarithmic expression.

Two sounds of identical (objective) intensities but with different frequencies are sensed by the human ear as different (subjective) intensities. As the *loudness* is frequency dependent, it can be determined by the following way: if the ear senses the intensities of a

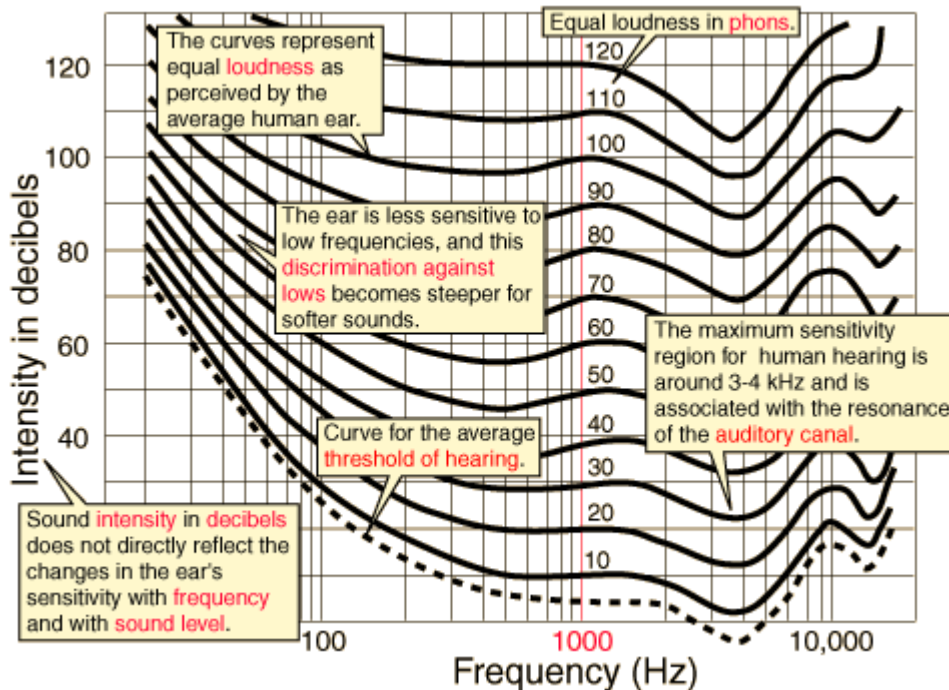
sound of arbitrary frequency (or arbitrary composition of frequencies) and of a 1 kHz sound of intensity I identical, then the *loudness* of the sound is defined as

$$H = 10 \lg \frac{I}{I_0} \text{ phon.}$$

The reference is chosen in 1936 as the threshold of audibility at 1 kHz harmonic sound: $I_0 = 1 \cdot 10^{-12} \text{ W/m}^2$. At this limit, the loudness of the sound is 0 phon. The loudness of the sound that causes pain in the ear is 130 phon (threshold of pain). The phon scale of the loudness is nothing else than the decibel scale of the harmonic sound of frequency 1 kHz:

$$H (\text{phon}) = H_{1 \text{ kHz}} (\text{dB}).$$

Source of sound	Subjective sound intensity (phon)
lower limit of audibility	0
rustle of leaf	10
whisper	20
noise of silent street	30
normal talk	50
shout	80
close to howling of the lion	120
upper limit of audibility, threshold of pain	130



How the speed of the sound depends on the properties of the medium?

In homogeneous and isotropic solid medium of infinite size, both longitudinal and transverse waves can propagate with speeds of

$$c_{\text{long}} = \sqrt{\frac{E}{\rho} \cdot \frac{1-\mu}{(1+\mu)(1-2\mu)}} \quad c_{\text{trans}} = \sqrt{\frac{E}{\rho} \cdot \frac{1}{2(1+\mu)}}$$

Here E is the Young's modulus (see the Hooke's law of the elasticity), ρ is the density and μ is the so called *Poisson's number* that expresses the transverse (cross) compression ($\Delta d/d$)

upon (longitudinal) elongation ($\Delta l/l$): $\mu = -\frac{\Delta d/d}{\Delta l/l}$. The Poisson's number is between 0 and

$\frac{1}{2}$, typically between 0.3 and 0.4. The ratio of the speeds of the longitudinal and transverse waves depends on the Poisson's number only:

$$\frac{c_{\text{long}}}{c_{\text{trans}}} = \sqrt{\frac{2(1-\mu)}{1-2\mu}}.$$

For many substances $\mu \approx 1/3$, therefore $c_{\text{long}} \approx 2 \cdot c_{\text{trans}}$. Generally, as $\mu \leq 1/2$, the longitudinal waves propagate faster than the transverse waves in the same solid medium. That can be utilized to localize the epicenter of an earthquake.

Example: When an earthquake occurs, both the longitudinal (compressional „P”) and transverse („S”) seismic waves radiate outward from the focus through the Earth’s crust. The P-waves travel faster ($c_P = 8.5$ km/s) than the slower S waves ($c_S = 4.7$ km/s). If a seismic recording station detects the arrival of the P-waves 3.5 minutes before the S-waves from a distant earthquake, how far away was the focus of the earthquake?

Solution: The distance traveled by the two waves is the same:

$$D = v_P t_P = v_S t_S.$$

The S-wave takes 3.5 minutes = 210 s longer to cover this distance:

$$t_S = t_P + 210 \text{ s}.$$

Substitution gives numerically $D = 2200$ km.

Special case: In (infinitely) long and (infinitely) thin elastic solid rod, transverse wave is not produced, thus longitudinal wave propagates. As there is no cross-compression ($\mu = 0$), the expression of the speed of the longitudinal wave reduces to:

$$c_{\text{long}} = \sqrt{\frac{E}{\rho}}.$$

Formally, very similar expressions are valid in fluids:

$$c = \sqrt{\frac{K}{\rho}},$$

where K is the *compression modulus* of the fluid: $K = -\frac{p}{\Delta V/V}$, i.e. the ratio of the pressure (p) and the relative volume change ($\Delta V/V$) generated by the pressure, and in gases:

$$c = \sqrt{\frac{\kappa \cdot p}{\rho}} \quad \text{for perfect gases} \quad c = \sqrt{\kappa \cdot RT},$$

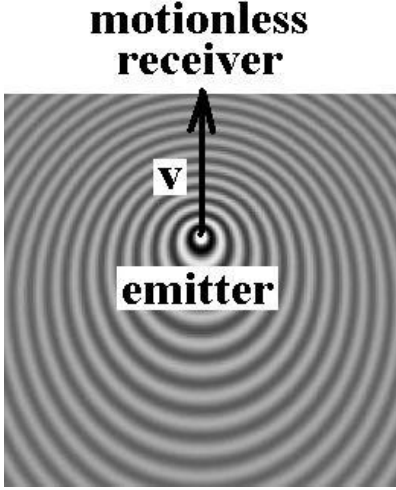
where $\kappa = c_p/c_v$ is the ratio of the *specific heat capacities* of the gas under constant pressure and volume, respectively, R is the universal gas constant and T denotes the absolute temperature (*Laplace’s expression*, 1816). In comparison, E (in solid states) can be formally replaced by K (in fluids) or by $\kappa \cdot p$ (in gases) in the expressions of the speed of longitudinal waves.

The Doppler-effect. The relative motion of the source (emitter) and the receiver influences the observed frequency. The phenomenon occurs in all types of waves (light, sound etc.) but the most common consequences can be experienced in the acoustics. For the sake of simplicity, we will assume that during the movement, the source and the receiver remain always on the same straight line. All velocities will be referred to the resting medium.

a) The source is resting and the receiver is moving. If the receiver is moving towards the resting source with velocity v_r , then it will detect not only f_0 vibrations within 1 s but additionally $v_r/\lambda_0 = v_r \cdot f_0/c$ more vibrations. Therefore, the approaching (+) or receding (-) receiver will observe the emitted f_0 frequency as

$$f = f_0 \left(1 \pm \frac{v_r}{c} \right)$$

frequency. The ratio v/c is called the *Mach number*. For example, at speed of approach (removal) of $v_r = \frac{1}{2} c$ (the Mach's number is $\frac{1}{2}$), the frequency of the observed sound will double (half), *i.e.* the pitch level increases (decreases) by one octave.



b) The source is moving and the receiver is at rest. The source moving with speed v_s to the receiver will emit the first phase of the oscillation at $t = 0$ and the last phase at $t = T_0$ when the source gets closer to the receiver by a distance $v_s \cdot T_0$. Therefore, the wavelength becomes shorter by $v_s \cdot T_0$ in front of the receiver. The new wavelength is $\lambda_0 - v_s \cdot T_0$ (see the figure). Because the shorter waves propagate in the resting medium with unchanged speed (c), the observed frequency is $f = c/(\lambda_0 - v_s \cdot T_0)$. If the source of sound approaches to (-) or moves away from (+) the resting receiver, the observed frequency can be given in the following expression:

$$f = \frac{f_0}{1 \mp \frac{v_s}{c}}$$

Note that the expressions obtained in special cases a) and b) are not symmetric to the speeds v_s and v_r , *i.e.* the observed frequency is not determined by the relative speed of the emitter and the receiver. This is in contrast to the optical Doppler effect (see later for explanation). The medium plays crucial role in this observation.

The combined expression is:

$$f = f_0 \frac{1 - \frac{v_r}{c}}{1 - \frac{v_s}{c}},$$

where v_s and v_r are the projected speed vectors of the source S and the receiver R to the interconnecting straight line SR . Both velocities should be measured relative to the medium. If the projected component of the vector shows to the $S \rightarrow R$ direction, then the sign is positive, if it is opposite, then the sign of the speed component is negative. Of course, if the SR distance remains constant (e.g. the movement is perpendicular to this direction), then the motion does not result in Doppler-shift of the frequency.

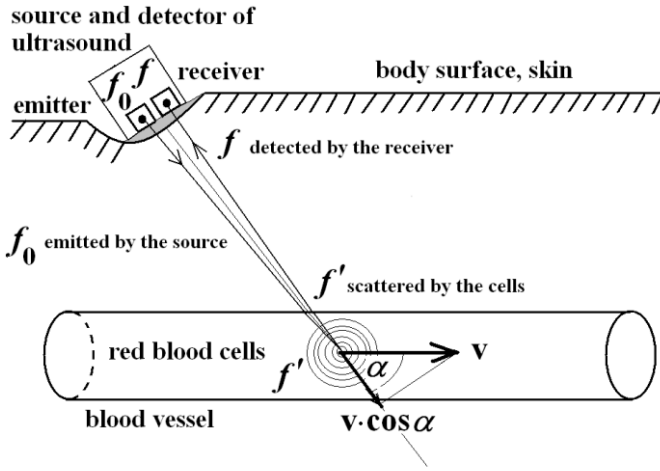
The physical background of the optical Doppler-shift differs from that of the acoustic version because no medium (no “ether”) is needed to mediate the optical waves (light). In the optical Doppler effect the medium plays no role and therefore the Doppler shift is determined by the relative speeds of the emitter and the receiver. The observed frequency can be calculated from the relativity theory (more precisely from the Lorentzian transformation):

$$f = f_0 \frac{1 \pm \frac{v}{c}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \quad \text{if } v \ll c, \quad \text{then } f = f_0 \left(1 \pm \frac{v}{c} \right),$$

where the sign of the velocity is positive if the distance between the source and the receiver is decreasing. For velocities much smaller than the speed of light in vacuum ($c = 3 \cdot 10^8$ m/s), the relativistic correction can be neglected and the expression derived from the classical physics is obtained.

Blood velocity measurement based on Doppler shift of ultrasound.

The tiny transducer including both the source and receiver of the ultrasound is placed on the



skin with a drop of a coupling medium above the blood vessel where the velocity of the red blood cells should be measured. The actual velocity vector of the red blood cell closes angle α with the straight line that connects the particle and the transducer. The projected component of the vector in this direction is $v \cdot \cos \alpha$ which is the speed of arrival or departure of the two objects (depending on the sign of the component). The wave of ultrasound of frequency f_0 radiated by the emitter is observed as frequency f' by the red

blood cell:

$$f' = f_0 \left(1 - \frac{v \cdot \cos \alpha}{c} \right).$$

The red blood cell scatters the ultrasound in all directions and detected by the receiver of the transducer as frequency f :

$$f = f' \frac{1}{1 + \frac{v \cdot \cos \alpha}{c}}.$$

From the two equations f' can be eliminated and the *Doppler-shift* (the difference of the frequencies of the received and the emitted radiations) can be expressed:

$$\Delta f = f - f_0 = -2f_0 \frac{\frac{v \cos \alpha}{c}}{1 + \frac{v \cos \alpha}{c}}.$$

As the velocity of the red blood cell (under all conditions) is orders of magnitude smaller than the speed of the ultrasound ($v \ll c$, the Mach number is $\ll 1$), the denominator is ≈ 1 :

$$\Delta f = -2f_0 \frac{\cos \alpha}{c} \cdot v,$$

which means that the Doppler-shift is linearly proportional to the velocity of the red blood cell and the proportionality factor depends on the directional term $\cos \alpha$. If the direction of the ultrasound is exactly perpendicular to the actual velocity of the red blood cell ($\alpha = 90^\circ$), then no Doppler-shift will be observed. The Doppler-shift reaches maximum if the direction is tangential ($\alpha = 0^\circ$), *i.e.* the vector of speed and the direction of the radiation are coinciding.

Optimum of the frequency of the ultrasound to measure the velocity of the blood.

It was shown that the Doppler-shift is proportional to the frequency of the ultrasound. The higher is the frequency, the larger will be the Doppler-shift and the more precise will be the determination of the velocity of the red blood cell:

$$\Delta f = \text{const}_1 \cdot f.$$

Unfortunately, the intensity of the reflected wave (echo) will show opposite tendency. In soft tissues and in frequency range (2-20 MHz) used in medical diagnostics, the loss factor α (the

sum of the absorption, scattering, etc.) in the exponent of the exponential extinction law increases proportionally with increase of the frequency:

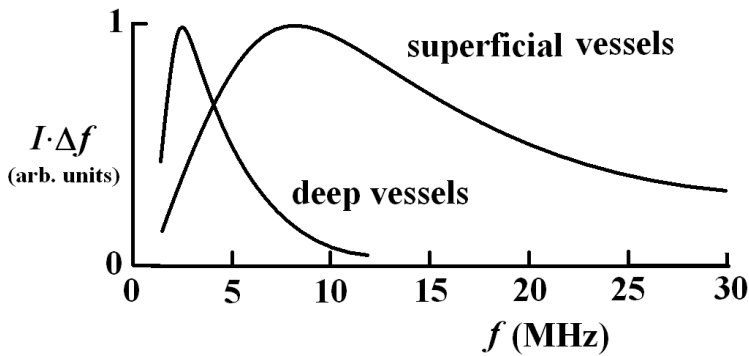
$$\alpha = \text{const}_2 \cdot f.$$

The intensity of the echo from a red blood cell from depth d is

$$I = \text{const}_3 I_0 \exp(-\alpha \cdot 2d),$$

where I_0 is the intensity of the emitted (incident) ultrasound which has to cover a distance $2d$ back and forth. We will consider the frequency as optimum (f_{opt}) if the $I \cdot \Delta f$ product is the largest (maximum). The necessary condition is the disappearance of the first derivative of $I \cdot \Delta f$ according to the frequency f : $d(I \cdot \Delta f)/df = 0$, which gives

$$f_{\text{opt}} = \frac{1}{2d \cdot \text{const}_2}.$$

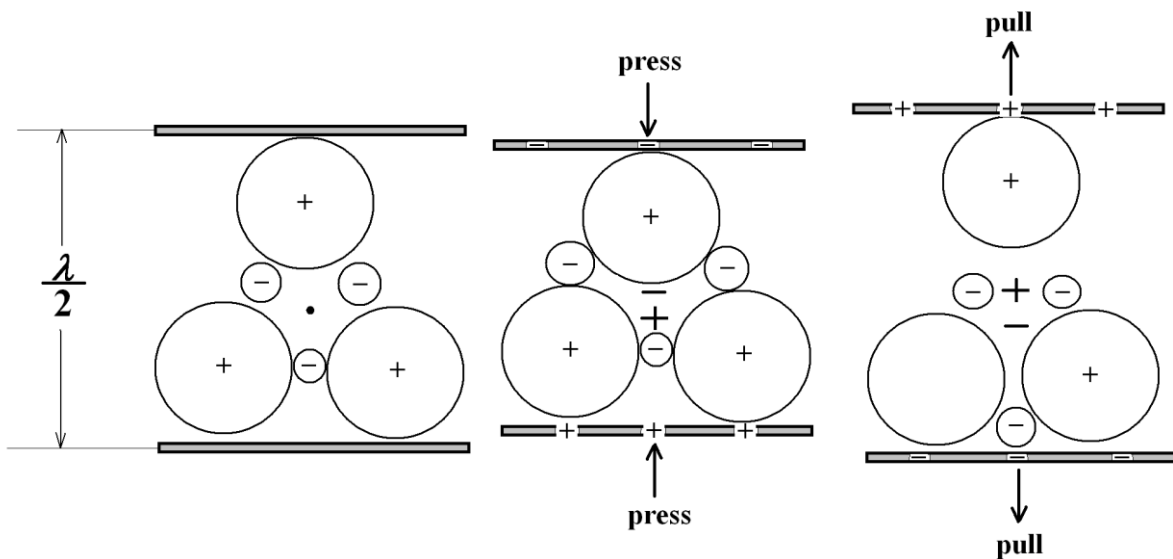


The figure offers impression how the $I \cdot \Delta f$ product behaves as a function of the frequency for two types of blood vessels. The “const” values were taken from experiments and were kept fixed. Both curves run through maxima whose frequencies are different: smaller for deep vessels and larger for superficial vessels.

In the latter case, the (half) width of the curve is much larger than in the former case. Therefore the selection of the optimum frequency is less crucial in investigation of the superficial vessels than the deep vessels. In practice, the expression

$$f_{\text{opt}} = \frac{90 \text{ MHz} \cdot \text{mm}}{d}$$

has proved to be convenient: 2-3 MHz frequency is used to study blood vessels in depths 3-4 cm (e.g. descending aorta) and 5-10 MHz frequency to investigate superficial blood vessels.



Production of ultrasound occurs mainly by inverse piezoelectric phenomenon. In piezoelectric crystals (e.g. lead-zirconate-titanate (PZT), synthetic ceramics, etc.), the position of the center of charges of the molecules will change upon mechanical deformation due to

pressing or pulling the crystal. In inverse case, the crystal can be forced to be pressed or to be pulled by putting electric potential to the electrodes attached to the surface of the crystal. If the polarity of the potential is alternating with frequency f , then the crystal will perform forced oscillation with the same frequency. If this frequency coincides with the self frequency of the crystal, then the amplitude of the vibration will be enhanced by resonance. A layer thickness of $\lambda/2 (= c/2f)$ will assure the condition of resonance: the fundamental frequency will be generated by standing wave with nodes at the electrodes.

Medical applications of the ultrasound. Depending on the intensity of the ultrasound, it can be used in different fields, among others in

- diagnostics: iconography (Be precautions! The higher is the resolution of the image, the larger intensity (or dose) is required.),
- therapy: mechanical effects $I > 0.1 \text{ W/cm}^2$ (cavitation, removal of gases from fluids, dispersion, breaking chromosomes $I > 1 \text{ W/cm}^2$),
thermic effects (see homework problem #19),
chemical effects (depolymerization, bleaching of dyes) etc. and
- surgery and urology: shock waves (*e.g.* breaking up kidney stones).

Potential dangers: although the ultrasound as mechanical wave used in medicine is generally considered as harmless radiation (“non-invasive” method), care should be taken because

- the threshold values for safe handling have poor scientific support, and are derived mainly from loose experiences and common wisdom and
- the danger of the accumulation of doses in the body cannot be definitely excluded (the similar danger is well known in the field of ionizing radiation).

The physical characteristics, thresholds and fields of applications are summarized in the table below.

$f = 1 \text{ MHz}$ maximum values in water	$I = 10 \text{ mW/cm}^2$ DIAGNOSTICS	$I = 3 \text{ W/cm}^2$ THERAPY	Shock waves
displacement: $x = \frac{\sqrt{2I/Z}}{\omega}$	2 nm in the range of thickness of membranes holding biomolecules in strict hierarchy	35 nm much larger than the thickness of membranes	Practically, the high intensity shock waves (pulses) consist of one half waves only, therefore the expressions derived for harmonic (continuous) waves cannot be applied without restrictions.
relative elongation: $x \cdot \frac{2\pi}{\lambda} = \frac{\sqrt{2I \cdot Z}}{E}$	$8,4 \cdot 10^{-6}$	$1,47 \cdot 10^{-6}$	
acceleration: $x \cdot \omega^2 = \omega \sqrt{\frac{2I}{Z}}$	$2 \cdot 10^3 \text{ m/s}^2$ (!) the gravitation constant is $g = 10 \text{ m/s}^2$	$3,5 \cdot 10^4 \text{ m/s}^2$ (!) the gravitation constant is $g = 10 \text{ m/s}^2$	
pressure of sound: $E x \frac{2\pi}{\lambda} = \sqrt{2I \cdot Z}$	$2 \cdot 10^4 \text{ Pa}$	$3,5 \cdot 10^5 \text{ Pa}$ this is the pressure of the tires of the car	

Summary of (absolute) basic definitions and expressions

1. A *wave* is a disturbance that moves through a medium, carrying energy but not matter.
2. The *amplitude* of a wave is defined as the maximum displacement from equilibrium that occurs during any cycle of the wave.
3. In *longitudinal* waves, the particles in the medium oscillate parallel to the direction of the wave propagation.
4. In *transverse* waves, the particles in the medium oscillate perpendicular to the direction of the wave propagation.
5. The *wavelength* of a wave is the shortest distance between two points of the same phase.
6. The *period* of a wave is the length of time it takes a wave to travel a distance equal to its wavelength. Equivalently, it is the time required for any particle in the medium to complete one oscillation as the waves pass.
7. The *frequency* of a wave is defined as the number of waves passing any fixed point per unit time. Equivalently, it is the number of oscillations per unit time completed by any of the particles in the medium.
8. A *simple harmonic wave* is one in which the oscillations that make up the wave are simple harmonic oscillations. These give rise to sinusoidal wave functions.
9. The *wave number* is defined as 2π radians, divided by the wavelength: $\bar{\lambda} = \frac{2\pi}{\lambda}$.
10. The *intensity* of a wave is defined as the power crossing a unit area oriented perpendicular to the wave velocity.
11. The *superposition principle* states that when two waves meet, each continues, undisturbed by the presence of the other, with its original velocity. Where the two waves are both present, their resultant is the point-by-point algebraic sum of the individual waves' displacements.
12. *Constructive interference* results when two waves arrive at the same point in phase with each other. In this case, the amplitude of their resultant is the sum of the component amplitudes.
13. *Destructive interference* results when two waves arrive at the same point 180° out of phase with each other. In this case, the amplitude of their resultant is the difference of the two component amplitudes. If the component waves have equal amplitudes, total cancellation can occur.
14. *Dispersion* is the name given to the property of certain materials in which the wave speed depends on the frequency of the waves: $c = c(f)$.
15. When two waves travel in the same direction with slightly different frequencies, their resultant has an amplitude that oscillates in time with a frequency equal to the difference of the two frequencies. The oscillating loudness (for sound waves) is known as *beats*.
16. The *intensity level* of a sound wave is measured in decibels (dB), defined by $10 \cdot \log \frac{I}{I_0}$,
where I_0 is an arbitrarily chosen reference intensity of $1 \cdot 10^{-12} \text{ W/m}^2$.
17. A *standing wave* is the result of superimposing identical waves traveling in opposite directions. It consists of a series of oscillating loops, separated by stationary points of zero motion, where the two component waves always cancel.
18. The points of no motion in a standing wave are called *nodes*.
19. The location of the maximum-amplitude oscillations in a standing wave are known as *antinodes*.
20. When a source of waves moves through the medium, or when the detector of the waves moves, a change in frequency and wavelength is observed. This frequency or wavelength shift is known as the *Doppler effect*.

21. A *shock wave* is a large-amplitude wave caused by a wave source moving through a medium faster than the speed of wave propagation.
22. The *Mach number* is the ratio of the speed of a wave source to the speed of sound in that medium. In medical practice, the Mach number is much smaller than 1 ($\sim 1\text{‰}$).

Suggested texts to consult

- J. J. Braun: Study Guide: Physics for Scientists and Engineers, HarperCollinsCollegePublishers, New York 1995 or any other college physics texts.
- P. Maróti, I. Berkes and F. Tölgyesi: Biophysics Problems. A textbook with answers, Akadémiai Kiadó, Budapest 1998.
- S. Damjanovich, J. Fidy and J. Szöllösi (eds.): Medical Biophysics, Medicina, Budapest, 2009.

Problems for home works and/or seminars.

- 1) How much is the wavelength of the normal tone “a” („from Vienna”, 440 Hz) in the air and in the water? Give the similar values for the “Hungarian” tone “a” (435 Hz)! What would be the experience of the audience if the orchestra tuned their instruments to these different notes?
- 2) The octave of a “wohl-temperiertes Klavier” consists of 12 notes whose frequencies follow a geometrical series (the ratios of the neighboring frequencies are equal). The evenly tuned scale was introduced in the music by the Bach’s works from 1720. How much is the frequency of the tone “c” if the scale is tuned to the normal tone “a” (440 Hz)?
- 3) What are the fundamental frequencies of the open and closed pipes of length 80 cm?
- 4) What frequencies will be specifically amplified in the external auditory canal of the human ear of length 2.5 cm? Does this effect increase or decrease the threshold of hearing?
- 5) The whales are very sensitive to underwater waves of low frequency. How long could be their external auditory canal if the maximum of hearing sensitivity is 100 Hz?
- 6) A porpoise sends an echolocating pulse (60 kHz) as it tracks the path of a shark. The power of the pulse is 30 mW. The intensity of the pulse at the position of the shark is $1.5 \cdot 10^{-5} \text{ W/m}^2$. (a) What is the distance between the shark and the porpoise? (b) What is the displacement amplitude of the water molecules adjacent to the shark?
- 7) Dogs have very sensitive hearing. Suppose their threshold of hearing is $1 \cdot 10^{-15} \text{ W/m}^2$. If a sound is judged by a human to be 50 dB, what is the correct dB rating for a dog?
- 8) Three loudspeakers each of 20 dB intensities are operating at the same time. How much is the resultant sound intensity?
- 9) The middle ear amplifies the pressure of sound by 20 times (the intensity of the sound by 400 times) during transmission from the ear drum of the external ear to the oval window of the inner ear. What would be the increase of the hearing threshold (in dB) if the middle ear failed this function?
- 10) A train is running through the railway station with 10 m/s speed. The frequency and intensity of the steam whistle of the locomotive are 1 kHz and 100 dB, respectively. Standing at the platform 1 m away from the rail, what would be the drop of intensity and observed frequency (pitch of the tone) 5 s after the locomotive passed by?
- 11) A bat emitting cries at 80 kHz flies directly at a wall. The frequency it hears is 83 kHz. How fast is it flying?
- 12) What is the radial speed of a star whose spectrum shows the wavelength of the Na spectral line of 589.6 nm at 592.0 nm? The speed of the light in vacuum is $3 \cdot 10^8 \text{ m/s}$.
- 13) Two sound waves of equal amplitudes propagate in the same direction. Their wavelengths in the air are 72.0 cm and 77.2 cm. Do we observe beats?

- 14) A circular ultrasound transducer of 1 cm diameter is operating in water at 1 MHz frequency. How much will be the diameter of the ultrasound beam at 4 cm from the transducer?
- 15) What is the angle of refraction of the ultrasound that arrives at 12° angle of incidence to the boundary of air ($c_{\text{air}} = 343 \text{ m/s}$) and muscle tissue ($c_{\text{muscle}} = 1590 \text{ m/s}$)?
- 16) What is the ratio of reflection and transmittance of the ultrasound at the boundary of muscle tissue ($Z = 1.7 \cdot 10^6 \text{ kg} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$) and fat ($Z = 1.35 \cdot 10^6 \text{ kg} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$)?
- 17) Do the air bubbles in water collect or disperse the (parallel) ultrasound beams? Do they act as converging or diverging lenses?
- 18) Estimate the penetration depths of the ultrasound of frequency 1 MHz in lung ($\alpha = 7 \text{ cm}^{-1}$), in bones ($\alpha = 3 \text{ cm}^{-1}$), in muscles ($\alpha = 0.3 \text{ cm}^{-1}$) and in blood ($\alpha = 0.03 \text{ cm}^{-1}$)! The sum of the losses due to absorption, scattering etc. (total extinction coefficient) is denoted by α , and the actual values are in the brackets.
- 19) A 10 cm depth section of the liver is investigated by ultrasound of frequency 1 MHz and intensity 1 W/cm^2 . The radiation lasts for 10 s. Estimate the temperature increase of the area! The sum of the loss coefficients in the liver is $\alpha = 0.17 \text{ cm}^{-1}$, and replace the liver by water (from thermal points of view) of 4.2 J/gK specific heat capacity.
- 20) The proper position of the plastic lens after cataract removal can be checked by ultrasound-echo experiment ("A"-image). The transducer is attached to the cornea and the echo from different layers of the eye bulb is monitored on the screen of an oscilloscope. The following signals can be visualized: „A” – initial echo that originates from reflection from the contact fluid between the transducer and the cornea, „B” – double echo that comes from the two boundaries of the cornea (difficult to separate), „C” and „D” – echo from the two surfaces of the lens and „E” – echo from the back wall of the eye bulb. How much is the length of the bulb if the (time) gap between the „B” and „E” echo amounts to $30 \mu\text{s}$? The speed of the sound is 1600 m/s .
- 21) The cataract is emulsified by low frequency (23 kHz) and large intensity (1 kW/cm^2) ultrasound (produced by magnetostriction) followed by drawing through a cut made between the cornea and the sclera. What should be the amplitude of the ultrasound? The acoustic impedance of the cataract is $1.75 \cdot 10^6 \text{ kg} \cdot \text{m}^{-2} \cdot \text{s}^{-1}$.