Mechanical vibrations (oscillations)
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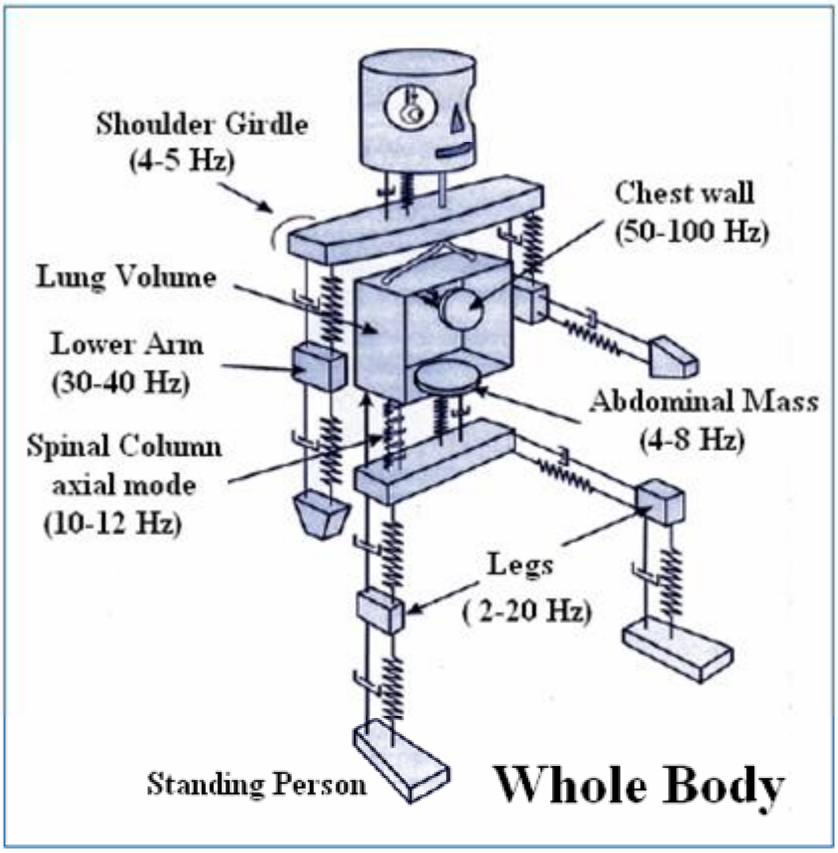
Every day's experience is the production, propagation and damping of wide variety of mechanical vibrations in human body. As early as in 1761, *L. Auenbrugger* introduced the traditional method of *percussion* (*sounding*) in the medical investigation. *A Hamilton* (1918) carried out the first systematic investigations among workers in the stone-quarry breaking solid rocks with pneumatic hammer driven by compressed air. Athletes jumping up and down in competitive way (*e.g.* basketball or handball players) may suffer severe consequences (damage) due of vibrations evoked by large forces in their bodies. Similar mechanical vibrations (that lead to shock waves) are produced in the eye bulb if the breaking power of the cornea is designed to modify ("throw away your glasses") by a series of high energy (UV) laser light pulses (graduate ablation (evaporation) of the layers of the cornea by intense laser flashes). It is called "sculpture of the cornea". Similar physical phenomena arise when the head of a boxer is hit by a strong and sudden punch (*e.g.* KO). The liquid brain closed in the solid skull (cranium) is exposed to heavy oscillations and may suffer severe damages.



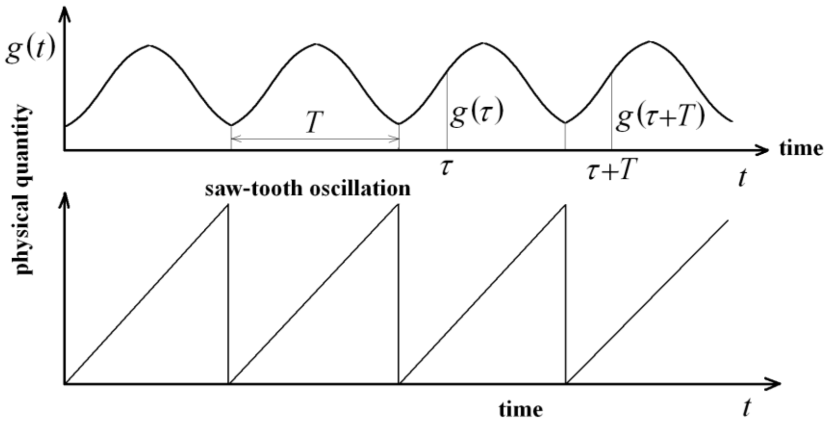
The oscillations can be classified as *driven oscillations* (the body or part of the body are exposed to external and periodic force) and *self* (own) *oscillations* (the body or part of the body are free to oscillate). The majority of the organs of the human body have low frequencies of self oscillations (see the figure) which are below the hearing (sensing) frequency limit of the human ear (infra oscillations that evoke infra sounds of frequencies smaller than 20 Hz). Mechanical external vibrations of low (infra) frequencies may cause damage in the human body without evoking alarming reactions by the hearing. The effect may be enhanced by *resonance* of the organ which has identical self frequency as the external periodic mechanical vibration.

The different organs are in mechanical coupling with each other. The complex set of mechanical vibrations can be described by an extended *network* of elastic (spring) and damping elements. The goal of the present lecture is to introduce you into the world of interacting vibrations that may exert significance consequences to the function of the human body.

Principal definition: a physical quantity makes *oscillation* (in strict (mathematical) sense of the word) if its value is periodic function of the time *t*:

$$g(t) = g(t+T) .$$

Here $g(t)$ denotes the actual value of the physical quantity at time *t*. The *time of period T* is the shortest time interval after which the physical quantity takes the same value as it had at time *t*. Its reciprocal (inverted) value is called *frequency*: $f = 1/T$, its dimension is 1/time and its unit is 1/s = 1 Hz. For example, if the heart beats 50 in a minute, then its frequency is $f = 50\ (\mathrm{min})^{-1} = 50/60$ Hz and its time of period is $T = 60/50$ s.



More generally, we talk about oscillations (vibrations) even in the lack of strict periodic changes of the physical quantity *versus* time. If the periodic character of the physical quantity can be recognized, the motion is frequently called as oscillation. For example: the damped oscillation does not belong to the cathegory of oscillations in the strict sence of the word (it is not periodic), however, we call it as oscillation because of the swinging character of the motion.

Classification: according to its actual mathematical form, the function $g(t)$ may include several types of oscillations.

Simple harmonic oscillation (*sine/cosine oscillation*): $g(t) = A\cdot\sin(\omega t+\alpha)$,
where *A* denotes the *amplitude*, ω is the *angular frequency* ($\omega = 2\pi/T$), $\omega t+\alpha$ is the *phase* and α is the *initial phase*. (It is recommended to measure the phase here not in *degree*s but in *radian*s.) The simple harmonic oscillation is described by a sole sine function of the time.

Anharmonic oscillation: the physical quantity performs simultaneously finite (see *e.g.* the Lissajous-curves) or infinite (see the Fourier-theorem) numbers of harmonic oscillations. Typical example is the *saw-tooth vibration* (see the bottom part of the figure). If a slow and long inspiration is followed by a sudden expiration periodically then the chest volume describes a saw-tooth vibration.

Simple harmonic oscillation.

Consider a point of mass *m* that carries out harmonic oscillation along axis *x* with equilibrium position at $x = 0$.

Kinematic description.

Position (coordinate): $x = A\cdot\sin(\omega t+\alpha)$. From the position-time relationship, other kinematic characteristics can be derived.

Velocity: $\mathrm{v} = \dfrac{dx}{dt} = A\omega\cdot\cos(\omega t+\alpha)$, which has maximum while crossing the origin at times $t = 0, T/2, T, \ldots$ and disappears ($\mathrm{v} = 0$) at the turning points (amplitudes) of the oscillation at times $t = T/4, 3T/4, \ldots$.

Acceleration: $a = \dfrac{d\mathrm{v}}{dt} = -A\omega^2\cdot\sin(\omega t+\alpha) = -\omega^2\cdot x$, which is proportional to the deviation from the origin and shows in opposite direction therefore it is directed always to the equilibrium position (origin).

Examples. What is the maximum of the acceleration of the heart beating 120 in a minute (supposing harmonic motion of the heart)? $a_{max} = -\omega^2 A = (2\pi/T)^2 A = 1.58$ m/s^2, which is one sixth of the gravitational constant ($g = 9.81$ m/s^2).

Estimate the maximum of the acceleration of the medium transferring ultrasound of frequency 1 MHz and of amplitude $A = 10$ nm (the radius of the hydrogen atom is ~ 0.1 nm)! $a_{max} = -\omega^2 A = (2\pi f)^2 \cdot A = 4\cdot 10^5$ m/s^2, which is forty thousand times larger than the gravitational constant ($g = 9.81$ m/s^2). The best trained pilots can survive 5-10 g (whole body) acceleration for short period of time, only. Under these conditions, take care of the *cavitation*!

Dynamic description. Substitute the position-time function of the movement in the principal law of the dynamics (Newton's 2nd law):

$$F = m \cdot a = -m\omega^2 \cdot x = -k \cdot x$$

The force is directly proportional and is oppositely directed to the elongation (see the *Hooke*'s *law* of the elasticity: $\sigma = E\cdot\varepsilon$, where $\sigma = F/A$ is the *mechanical tension*, E denotes the*Young's modulus of elasticity* and $\varepsilon = \Delta l/l$ is the *relative elongation*). This is the dynamic condition of production of the simple harmonic oscillation.

Forces directed permanently to the stable equilibrium position will create oscillating motions. However, simple harmonic oscillation will be established only if the actual force is proportional to the elongation.

The proportionality factor in the expression of the force is called *directional force* (*spring-constant*):

$$k = m\omega^2 = m\left(\frac{2\pi}{T}\right)^2,$$

its dimension is force/distance and the unit is N/m.

The time of period of the harmonic oscillation is:

$$T = 2\pi\sqrt{\frac{m}{k}},$$

i.e. the time of period is directly proportional to the square root of the mass (if k is constant) and indirectly proportional to the square root of the directional force (if m is unchanged).

Energy of the mass under harmonic oscillation. The kinetic energy (½ mv^2) and the potential (elastic) energy (½ kx^2) of the mass oscillate in opposite phase: if the kinetic energy has a maximum, then the potential energy is at minimum and *vice versa*. Because the field of force is conservative, the total mechanical energy (the sum of the kinetic and potential energies) will remain constant. The energy is not dissipated:

$$E_{\text{total}} = \frac{1}{2}m\text{v}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 A^2 = \text{constant}.$$

If the potential energy (½ kx^2) is plotted against the elongation (x) in a rectangular (Descartes) coordination system then parabola with upward oriented branches will be obtained. The total energy of the mass with 2 times, 3 times, etc. larger amplitude will be 4 times, 9 times, etc. larger (if the frequency remains constant). Similar quadratic relationship is valid for the frequency (if the amplitude remains constant).

Creation of harmonic oscillation. The spring produces elastic force which is proportional to the elongation and shows toward the equilibrium position. Therefore, a mass attached to a spring carries out harmonic oscillation. At small amplitudes, both the mathematical and the physical *pendulum* make harmonic oscillations.

Superposition of harmonic oscillations. A point under the influence of two independent effects (forces) can undertake two independent oscillations simultanously. The vibrations can be summed up without mutual disturbance. The general superposition of two arbitrary linear oscillations can be traced back to two principal cases.

1. Addition (superposition) of two unidirectional vibrations.

a) The frequencies are equal.

$$x_1 = A_1 \sin \omega t$$
$$x_2 = A_2 \sin(\omega t + \alpha_0)\ .$$

The resultant oscillation is the simple (algebraic) sum of the components:

$$x = x_1 + x_2 = A \sin(\omega t + \alpha)\,,$$

where

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos\alpha_0} \text{ and } \mathrm{tg}\alpha = \frac{A_2 \sin\alpha_0}{A_1 + A_2 \cos\alpha_0}\,.$$

The resultant vibration has the same direction and frequency as the components and the amplitude A together with the phase constant α depend on the amplitudes A_1 and A_2 and the mutual phase constant α_0 of the components.

Special cases:

- If the phases are equal (the two component vibrations are "in phase", $\alpha_0 = 0$), then the amplitudes of the components are summed up: $A = A_1 + A_2$, and the resultant phase constant coincides with that of the components: $\alpha = 0$. This is called *constructive superposition.*
- If the vibrations have opposite phases ($\alpha_0 = \pi$), then the difference of the amplitudes should be taken: $A = |A_1 - A_2|$, and the resultant phase is equal to the phase of the component of larger amplitude. If $A_1 = A_2$, then $A = 0$, *i.e.* the two vibrations cancel (quench) each other (see later the similar effect of *interference* of waves of sound or light). This phemonenon is called *destructive superposition*.

b) The frequencies are different..

$$x_1 = A_1 \sin \omega_1 t$$
$$x_2 = A_2 \sin(\omega_2 t + \alpha_0)\ .$$

Now, the resultant oscillation cannot be taken to the form of $x = x_1 + x_2 = A \sin(\omega t + \alpha)$, consequently it is not a harmonic oscillation. In addition, it is even not a periodic motion. The resultant motion will be periodic only if the ratio of the frequencies of the two components (ω_1/ω_2) is a rational number. If this is the case, then $\omega_1 = n_1 \cdot \omega$ and $\omega_2 = n_2 \cdot \omega$ (n_1 and n_2 are relative prime integer numbers), and the values of the function $x = A_1 \sin(n_1 \omega t) + A_2 \sin(n_2 \omega t + \alpha_0)$ will be repeated in a time of period $T = 2\pi/\omega$.

Special case: superposition of two unidirectional harmonic oscillations with very close frequencies; the *beats*.

For the sake of simplicity, the amplitudes of the two oscillations are the same and the phase difference is zero:

$$x_1 = A \sin \omega_1 t$$
$$x_2 = A \sin \omega_2 t\ .$$

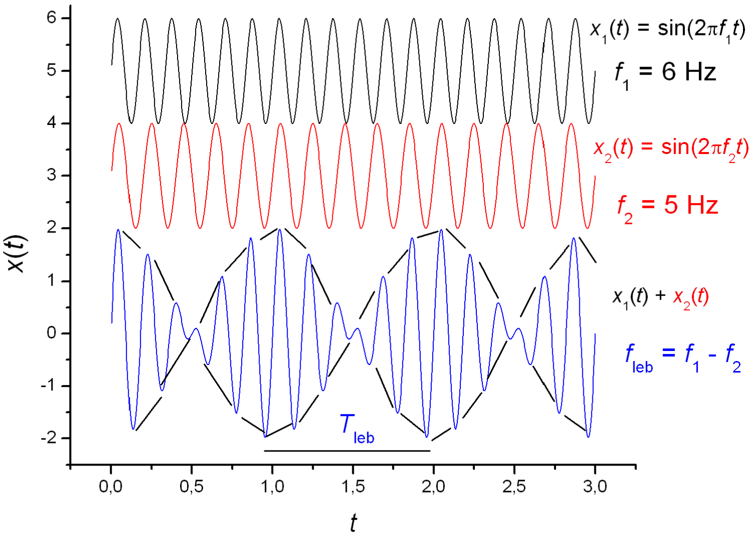
The resultant oscillation is

$$x = x_1 + x_2 = 2A \cos\frac{\omega_1 - \omega_2}{2} t \cdot \sin\frac{\omega_1 + \omega_2}{2} t\,.$$

As $\omega_1 \approx \omega_2$, the cosine factor changes with time much slower than the sine factor. The resultant vibration can be considered as a sine oscillation with angular frequency of $\frac{\omega_1 + \omega_2}{2}$, and amplitude of $2A \cos\frac{\omega_1 - \omega_2}{2} t$ that changes relatively slowly between $2A$ and 0. This phenomenon is called *beat* and the time elapses between two neighboring maxima of the amplitude is called *period of beat*, T_{beat}. This value is half of the period of the function $2A\cos\frac{\omega_1-\omega_2}{2}t$, (which is $\frac{2\pi\cdot 2}{\omega_1-\omega_2}$), *i.e.* $T_{\text{beat}} = \frac{2\pi}{\omega_1-\omega_2} = \frac{1}{f_1-f_2}$.

The frequency of the beat is $f_{\text{beat}} = 1/T_{\text{beat}}$, the difference of the frequencies of the two component oscillations:

$$f_{\text{beat}} = f_1 - f_2.$$



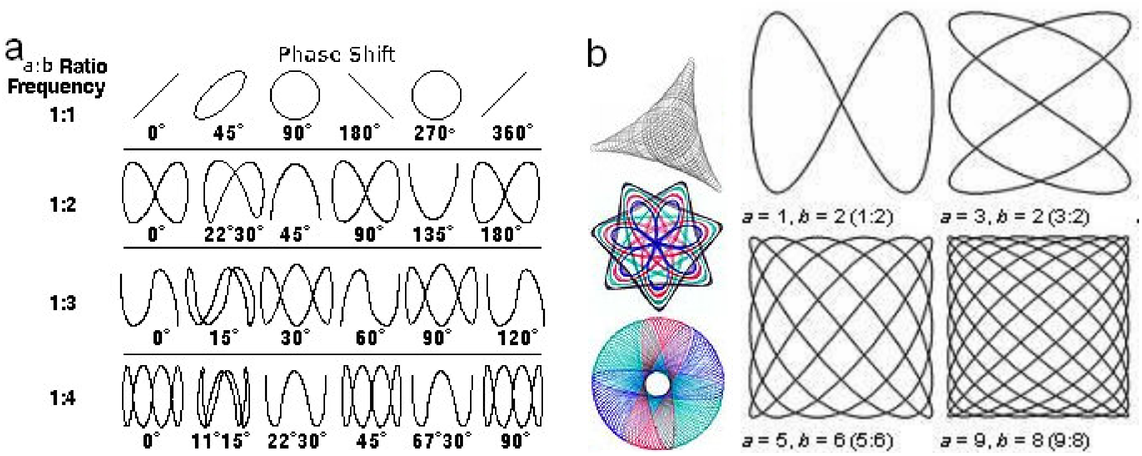
The phenomenon of beat is best and easiest to recognize in hearing (acoustics): if two tuning forks of frequencies 440 Hz and 446 Hz are oscillating simultaneously, then 6 pulsations (beatings) can be heard in a second. If the amplitudes are not equal, then the minima and maxima are less distinct.

2. Superposition of perpendicular oscillations, the *Lissajous-curves*.

a) The frequencies are the same. The resultant oscillation is generally "elliptic oscillation" (elliptically polarized oscillation):

$$x = A\sin\omega t \qquad y = B\sin(\omega t + \alpha),$$

which means that the oscillating point describes an ellipse in the *xy* plane. This can be proven if the common parameter, the time *t* is eliminated from the two equations. The size and position of the ellipse are determined by the amplitudes *A* and *B* and the phase difference *α* (see the first line of the figure).



b) The frequencies are different. Although both the x and the y coordinates change periodically as a function of time in the basic equations

$$x = A\sin\omega_a t \qquad y = B\sin(\omega_b t + \alpha),$$

the resultant motion is not necessarily periodic. It will be periodic only, if the ratio of the two frequencies (ω_a/ω_b) is a rational number (see the figure where ω_a/ω_b = ½, 1/3, ¼, 3/2, 5/6 and 9/8 and $A = B$). In opposite cases (e.g. if $\omega_a/\omega_b = \sqrt{2}$ which is an irrational number), the curves in the xy plane are not closed, *i.e.* the moving point will never return to the initial position.

Decomposition of the oscillations into harmonic oscillations; the Fourier theorem. We saw previously that the sum of harmonic oscillations whose frequencies were integral multiples of a fundamental frequency resulted in always periodic processes *i.e.* oscillations (however, not necessarily harmonic oscillations). The inverse problem may be interesting, as well: is it possible to decompose a given oscillation into the sum of harmonic oscillations? In the case of “normal” (smooth) functions which occur in the practice, the answer is a sound and unambiguous YES!

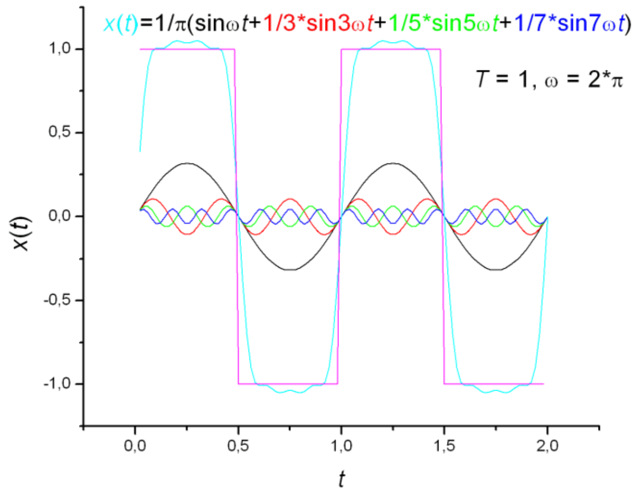
The Fourier-theorem: if $g(t)$ is a periodic function of the time, *i.e.* $g(t) = g(t+T)$, then it can be decomposed into the sum of sine and cosine functions in one way only where the amplitudes (A_i és B_i, i = 0,1,2,...) of the harmonics (components) are different and the frequencies $\omega_i = i\cdot\omega$ (i = 1,2,...) are integral multiples of the fundamental frequency ω:

$$g(t) = A_0/2 + A_1\cos\omega t + A_2\cos 2\omega t + ... + B_1\sin\omega t + B_2\sin 2\omega t + ...$$

where the coefficients (amplitudes) A_i and B_i are determined by the following integrals:

$$A_i = \frac{2}{T}\int_0^T g(t)\cos(i\omega t)\,dt \qquad B_i = \frac{2}{T}\int_0^T g(t)\sin(i\omega t)\,dt \qquad (i = 0,1,2,3,...).$$

In many practical cases, the function $g(t)$ is well approximated by only a few initial terms of the Fourier-series (see the example of the step function in the figure). Algorithms written to computers are available for fast Fourier transformation (FFT) of arbitrary periodic functions.



Example: the step function oscillation can be decomposed into the following set of harmonics with decreasing amplitudes:

$$x(t) = \frac{1}{\pi}(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + ...).$$

The sum of the first four terms of the series approximates the step function fairly well (see the figure where the contribution of the different terms is color coded).

The distribution of the amplitudes of the harmonics according the frequencies is called *spectrum* of the Fourier decomposition. The spectrum can be *discrete* (if the oscillation is strictly periodic and the Fourier-series can be applied) or *continuous* (in the case of aperiodic functions when the function is decomposed according to Fourier integral instead of Fourier series). The Fourier-transformation (FT) converts the function from the time space into the frequency space ($t \rightarrow \omega$) where different type of (spectral) analysis can be carried out. Typical application is the Fourier-transfomation of the infrared (IR) spectra (FTIR) based on which specific atomic and molecular vibrations can be detected and separated from millions of other vibrations.

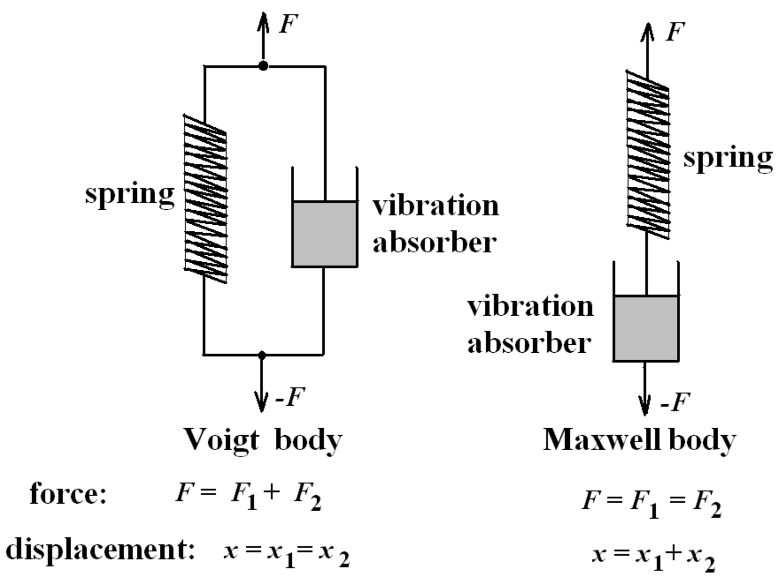
The most trivial application of the Fourier-theorem is the decomposition of sound vibration into harmonics (harmonic analysis). The sound is peeled into fundamental vibration whose frequency ω determines the pitch level (*intonation*) of the sound and into several harmonics oh frequencies $\omega_i = i \cdot \omega$ (i = 1,2,...) whose amplitudes determine the *tonality* of the sound.

Damped oscillations.

In living sytems, one can hardly find any manifestations of abstractions (*e.g.* point of mass, solid state, ideal gases and fluids) introduced in physics. Investigating the elasticity of real substances, they show generally both elastic (as an ideal spring) and viscous (as real fluids) properties. These substances are called *viscoelastic materials*. If they are forced to oscillate, their free oscillation will show monotonously decreasing amplitude. After several passes through the equilibrium position, the oscillation will be ceased soon or later (depending on the rate of energy dissipation via friction). If the damping is very large, the oscillation will perform half a period only, which means that the movement will stop even before passing through the equilibrium position (one sided "swinging" or *aperiodic threshold*).

In the simplest case, the models describing the oscillations of realistic substances consist of two components at least.

- *Linear spring*: upon elongation, the force is linearly proportional with and opposite to the displacement x: $F = -kx$. Therefore, the spring obeys the ideal (linear) law of the elasticity (Hooke's law).
- *Vibration absorber (damper)*: upon deformation, the (friction) force evoked is linearly proportional with and opposite to the actual velocity: $F = -\eta v$, where η is the viscosity of the fluid in the damper. The vibration absorber obeys the Newton's law of viscous fluids. The shock absorbers are essential and well known constituents of cars and airplanes. From the combinations of these two elements, several complex mechanical models can be constructed which can describe the principal



mechanical (kinematic and dynamic) behaviour of real substances under driven (forced) and free oscillations (see the introductory figure ("Xmas tree" model) of the wide range of oscillations of the whole or parts of the human body).

Here, we will deal with the simplest model only. In the *Voigt model*, the two elements (the spring and the damper) are coupled parallel. The displacements x are equal in the two branches but the forces are different: $F_1 = -kx$ in the branch of the spring and $F_2 = -\eta v$ in the branch of the absorber. The resultant force that acts on the mass m is the sum of the two components: $F = F_1 + F_2$. The equation of motion (Newton's 2[nd] law) is: $m \cdot a = -kx - \eta v$. Replacing the acceleration by $a = d^2x/dt^2$ and the velocity by $v = dx/dt$, the equation will be

$$m \cdot \frac{d^2x}{dt^2} = -k \cdot x - \eta \cdot \frac{dx}{dt}.$$

The solution of this differential equation, $x(t)$ can be given in closed (analytical) form that depends on the magnitude of the damping (friction) force. If it is smaller than a definite limit then the movement will be periodic (damped oscillation). If, however, the friction is larger than this limit, then the movement will not be anymore oscillatory but approaches monotonously to the equilibrium position (aperiodic movement).

Periodic movement: $\left(\frac{\eta}{2m} = \right)\kappa < \omega_0 \left(= \sqrt{\frac{k}{m}}\right)$. In this case, the vibration can be considered as harmonic oscillation whose amplitude decreases with time exponentially:

$$x = A \cdot e^{-\kappa t} \cdot \sin(\omega t + \alpha),$$

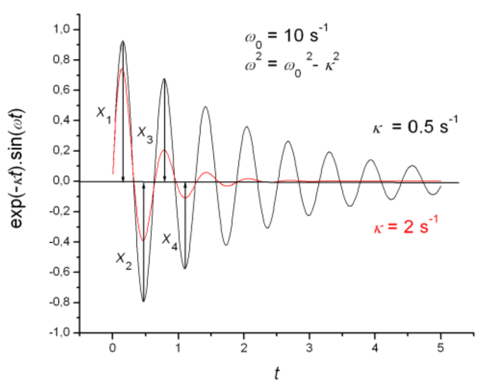
and the angular frequency of the vibration is

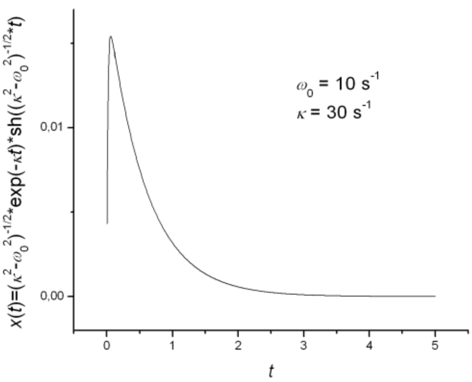
$$\omega = \sqrt{\omega_0^2 - \kappa^2},$$

where ω_0 is the angular frequency of the undamped ($\kappa = 0$) oscillation. Both the rate of the decrease of the amplitude and the actual angular frequency depend on the degree of the damping (κ). The larger is the damping (friction), the larger is the rate of loss of the amplitude and the smaller is the actual angular frequency. As the ratio of two neighboring maxima in the same direction is constant, it can be introduced as characteristics of the damping:

$$\frac{x_1}{x_3} = \frac{x_2}{x_4} = \ldots = \frac{x_n}{x_{n+2}} = e^{\kappa T},$$

where $T = 2\pi/\omega$.





Aperiodic movement: $\kappa > \omega_0$, *i.e*. the friction is larger than a definite threshold. The analytical solution of the equation of the movement at $x(t=0)=0$ and $v(t=0)=v_0$ initial conditions is

$$x = \frac{v_0}{\sqrt{\kappa^2 - \omega_0^2}} e^{-\kappa t} \mathrm{sh}\left(\sqrt{\kappa^2 - \omega_0^2}\, t\right),$$

which means that the deviation from the equilibrium position ($x = 0$) is always unidirectional (x remains negative or positive, depending on the initial conditions). During the movement, the mass never crosses the equilibrium position but approaches to it only while $t \to \infty$. This is a typical motion of a deviated pendulum in highly viscous fluid (*e.g.* in honey).

Forced (induced) vibrations. Resonance.
We have to make clear distinction between *free* and *forced oscillations.* After prompt deviation from the equilibrium state, the oscillating system left alone will carry out free (damped or undamped) vibrations as a result of internal forces (elastic force, friction etc.). If the system is exposed to periodic and external force (which will act not promptly but continuously), then the system will make forced (induced) vibrations. The dynamic equation of the motion can be created from that of the damped oscillation extended by the periodic external driving force:

$$m \cdot \frac{d^2x}{dt^2} = -k \cdot x - \eta \cdot \frac{dx}{dt} + F_0 \sin \omega t \,.$$

In the case of $\kappa < \omega_0$ (the damping is small), the general solution can be given in analytical form:

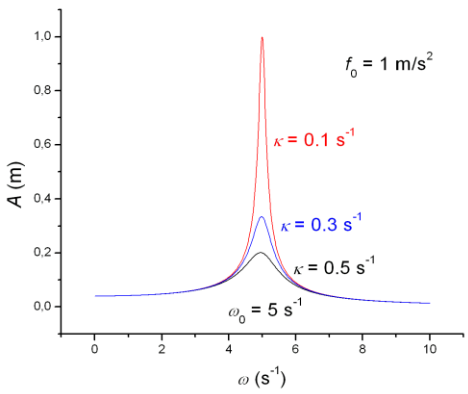
$$x(t) = A\cos(\omega t - \varphi) + ae^{-\kappa t} \sin\left(\sqrt{\omega_0^2 - \kappa^2} \cdot t + \alpha\right),$$

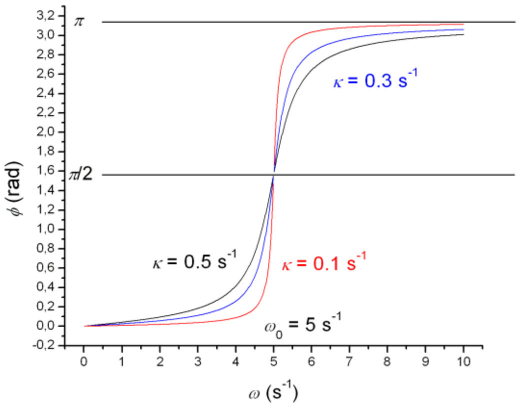
where

$$A = \frac{F_0 / m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\kappa^2\omega^2}} \qquad \mathrm{tg}\varphi = \frac{2\kappa\omega}{\omega_0^2 - \omega^2}\,,$$

and a and α are integration constants that can be determined from the initial conditions.

As can be seen from the position-time relationship, while the system follows the constraint (first term in the expression), it tries to preserve its own vibration (second term). The superposition of the two movements results in an undamped harmonic vibration of frequency ω and a damped self oscillation. The latter, however, will be terminated after short transient due to the damping and remains permanently the induced $A \cdot \sin(\omega t - \varphi)$ harmonic oscillation. The amplitude A and the phase difference φ between the driving force and the induced vibration depend on the angular frequency ω of the external force.





Starting from very low frequencies, the amplitude A increases gradually with increasing frequency and reaches maximum at $\omega = \omega_0$. The amplitude will decrease after further increase of the frequency. The amplitude of the induced vibration is at maximum if the frequency of the driving force coincides with the frequency of the self oscillation of the system. This phenomenon is called *resonance* and the $A = A(\omega)$ function is denoted as *curve of resonance*.

The smaller is the damping of the system κ, the sharper is the curve of resonance. In ideal case, the amplitude would tend to infinity in lack of any damping. This is called *catastrophe of resonance*.

Similarly interesting is the function of the phase shift vs. driving frequency. The phase difference monotonously increases from 0 to $\pi/2$ and from $\pi/2$ to π while the frequency increases from 0 to ω_0 and from ω_0 to infinity, respectively. Although the (forced) oscillating system takes over the frequency from the external constrain, it follows the driving force with certain delay (phase difference) only. In resonance, the phase delay is $\pi/2$, i.e. the $F_0 \sin \omega t$ driving force will be unidirectional to the actual velocity of the oscillating mass, $v = v_0 \cos \omega t$, which means that the external force can accelerate and offer energy to the mass with the highest yield. This is the background of the extreme features observed under (catastrophy of) resonance.

<u>Summary of (absolute) basic definitions and expressions</u>

1. The *amplitude* of an oscillation (A) is the maximum displacement from the equilibrium position.
2. The *period* of oscillatory motion (T) is the shortest time that elapses between successive occurrences of the same configuration.
3. The *frequency* of oscillatory motion (f) is defined as the number of oscillations that occur per unit time (this is also known as the *linear frequency*, to distinguish it from the *angular frequency* ω): $f = \frac{1}{T}$ (and $\omega = \frac{2\pi}{T}$).
4. *Harmonic motion* is a special type of oscillatory motion that results from a restoring force, F (or torque) which is directly proportional to the displacement from equilibrium (Δx) and directed always towards the equilibrium position: $F = -k \cdot \Delta x$. The proportionality factor, k is called *directional force constant.*
   Simple harmonic motion is characterized by solutions that involve the „harmonic functions" (sine and cosine) and a period that is independent of the amplitude.
5. The *phase constant* (α) is a constant in the argument of the sine (or cosine) function used to describe the oscillatory motion: $x = A \cdot \sin(\omega t + \alpha)$. It is determined by the initial state of the system.
6. A *simple* (or *mathematical*) *pendulum* consists of a point particle of mass m, swinging from a massless string of length l. The period T of small oscillations is independent of the mass and the amplitude, and depends only on the length of the string and the acceleration of gravity, g: $T = 2\pi\sqrt{\frac{l}{g}}$.
   If the rotational inertia of the pendulum differs from that of a point particle, we have a *physical pendulum*. If the object is pivoted at a distance L from its center of mass and allowed to swing with small amplitude, the period T of oscillations is $T = 2\pi\sqrt{\frac{I}{mgL}}$, where I is the moment of inertia for rotations about the pivot.
7. Oscillations with decreasing amplitude constitute *damped harmonic motion* during which dissipative forces (friction, fluid viscosity, etc.) decrease the amplitude of the oscillation with time because the mechanical energy of the system is gradually converted into thermal energy.
8. To maintain constant amplitude if damping forces are present, it is necessary to replenish the mechanical energy of the system. The resulting oscillations are known as *driven harmonic motion.*

9. If the frequency of the driving force matches the natural frequency of oscillation of the system, the system is in *resonance*. In resonance, the amplitude of the oscillations reaches a maximum and the phase shift between the oscillating driving force and the oscillating motion of the system is $\pi/2$ (= 90^o).

Suggested texts to consult

J. J. Braun: Study Guide: Physics for Scientists and Engineers, HarperCollinsCollegePublishers, New York 1995 or any other college physics texts.
P. Maróti, I. Berkes and F. Tölgyesi: Biophysics Problems. A textbook with answers, Akadémiai Kiadó, Budapest 1998.
S. Damjanovich, J. Fidy and J. Szöllősi (eds.): Medical Biophysics, Medicina, Budapest, 2009.

Problems for home works and/or seminars.

1) A basketball player of mass 100 kg lifts his center of mass by 1 m on jumping up. On landing, he needs 10 cm (elastic landing) or 1 cm (inelastic damping) path to damp completely his speed. Estimate the forces of damping (evoked in the vertebral column) on landing!
2) The mechanical role of the tiny bones in the human middle ear is approximated by the following model: a point mass of m = 2 mg is anchored to the ear drum and to the oval window by two springs of directional forces k_1 = 72 N/m and k_2 = 7.2 N/m, respectively. How much is the (self) frequency of this system?
3) The mass of an unloaded car is 800 kg. The body of the car will sink 6 cm after getting in 5 persons of total mass of 500 kg. How much is the time of period of vibration of the unloaded car and loaded with passengers?
4) A log floating on the surface of the water is pressed slightly down and left alone. Determine the frequency of the swinging log!
5) The walking can be considered as movement of the unloaded leg as physical pendulum from the back ahead in a passive way (without intervenience of the muskels of this leg). Estimate the speed of walking if the length of one foot step is s = 0.8 m and the unloaded leg of length l = 1 m is swinging around the hip as pivot axis!
6) Demonstrate the basic laws of superposition of vibrations using computer graphics methods! Construct simple Lissajous-curves!
7) The Fourier-decomposition of triangle-shaped vibration of unit amplitude is

$$x(t) = \frac{8}{\pi^2}(\sin\omega t - \frac{1}{3^2}\sin 3\omega t + \frac{1}{5^2}\sin 5\omega t - \frac{1}{7^2}\sin 7\omega t + ...)\ .$$

By plotting the (sum of the) components, demonstrate that a few components give acceptable approximation.
8) Give the equation of motion of a point mass attached to a spring and a damper coupled in series (Maxwell body)!
9) A mosquito is hitting with his leg the Szeged downtown bridge with a frequency identical with the self frequency of the bridge. In contradiction to the expectations of the catastrophe of resonance, the bridge will not collapse. Why not?
10) The springs of the wagons absorb periodic shocks at the connections of the rails and vibrations will be evoked. The compression of the springs is 1.6 μm upon 1 N load, the mass of the wagon is 22 tons and the length of the rails is 18 m. At what speed is the amplitude of the vibration the largest?