Flow of fluids 1

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Contents

Fluid mechanics in medicine

- circulation
  - arterial system
  - capillary system
  - venous system
  - lymphatic circulation
- respiration
  - upper respiratory tract
  - peripheral respiratory tract
- other bodily fluids

States of matter

	solid	liquid	gas
Cohesion	strong forces can turn repulsive	short-term molecular forces	no forces (ideal gas) weak forces (real gas)
Motion of particles	vibration	Brownian motion particles roll over each other	Brownian motion
Space-filling	constant volume	constant volume	fill up space
Shape	constant shape great forces needed to change it	no defined shape takes shape of container	no defined shape
Compressibility	repulsive forces prevent compression	repulsive forces prevent compression	compressible

What is a fluid?

- *'something that can flow'*
- *'something that has no defined shape'*
- ***fluid:*** a substance that cannot support shear stress
- deformation:
    - *tensile stress (tension)*
        - *'tends to change the dimensions but not the shape'*
        - force $\perp$ surface
        - $p = \frac{F}{A}$
    - *shear stress*
        - *'tends to change the shape but not the dimensions'*
        - force $\parallel$ surface
        - $\tau = \frac{F}{A}$
- *fluid* is a common name for *liquids* and *gases* — in most cases, they can be treated the same way

Pressure and density

- **pressure:** the ratio of the magnitude of the total force F exerted on a surface to the area A of the surface

$$p := \frac{F}{A}$$

- **density** of a substance: its mass per unit volume —

$$\varrho := \frac{m}{V}$$

- if the object is not homogeneous, we have to define density for a very small volume ΔV within which the density can be approximated as constant:

$$\varrho := \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{\mathrm{d}m}{\mathrm{d}V}$$

- the SI unit of density

$$\left[\varrho \right] = 1 \, \frac{\mathrm{kg}}{\mathrm{m}^3} = \frac{1000 \, \mathrm{g}}{10^6 \, \mathrm{cm}^3} = 10^{-3} \, \frac{\mathrm{g}}{\mathrm{cm}^3}$$

Classification of fluids

1. **Compressibility**
   - A. **compressible:** its density (ϱ) can change (eg, gases)
   - B. **incompressible:** its density (ϱ) is constant (eg, liquids)
2. **Internal friction (viscosity)**
   - A. **viscous:** there IS internal friction
   - B. **non-viscous:** internal friction is negligible

- *ideal fluid:* incompressible and non-viscous

Thermal equilibrium, temperature and heat

- **heat:** the energy associated with the random thermal motion of the particles of matter
- heat and temperature, eg: if we remove a metal ice tray and a cardboard box of frozen vegetables from the freezer, the ice tray feels colder than the box even though *both are at the same temperature* — metal transfers energy by *heat* at a higher rate than cardboard does
- two objects are in **thermal contact** with each other if energy can be exchanged between them by processes due to a temperature difference
- *thermal equilibrium* is a situation in which two objects would not exchange energy by heat or electromagnetic radiation if they were placed in thermal contact
- **temperature:** the property that determines whether an object is in thermal equilibrium with other objects
- **two objects in thermal equilibrium with each other are at the same temperature;** if two objects have different temperatures, then they are not in thermal equilibrium with each other

The basis of thermal expansion

- thermal expansion is a consequence of the change in the average separation between the atoms in an object
- as the temperature of an object increases, the average separation between them increases ⇒ the volume of the object increases

1. Figure: Expansion joint

Linear expansion

- if thermal expansion is sufficiently small relative to the initial dimensions of the object, the change in any dimension is, to a good approximation, proportional to the first power of the temperature change:

  $$\Delta L = \alpha L_i \Delta T,$$

  where $\Delta L = L_f - L_i$ is the change in a dimension with L_i being the initial dimension and L_f being the final dimension, and α denotes the **average coefficient of linear expansion**

- the coefficient of linear expansion depends on the material

- we can rewrite this equation to obtain the dimension of an object $L(T)$ at a temperature T if the dimension L_0 at some temperature T_0 is known:

  $$L(T) = L_0 + \Delta L = L_0 + \alpha L_0 \Delta T = L_0(1 + \alpha \Delta T)$$

- it can be shown that a cavity in a piece of material expands in the same way as if the cavity were filled with the material

Surface and volume expansion

- as linear dimensions of an object change with temperature, surface area and volume change as well
- surface change for a square:

$$A(T) = L(T)^2 = [L_0(1+\alpha\Delta T)]^2 = L_0^2(1+\alpha\Delta T)^2 = A_0\left[1+2\alpha\Delta T + \alpha^2(\Delta T)^2\right]$$

- which can be approximated as

$$A(T) = A_0(1+2\alpha\Delta T),$$

because α is usually very small and the terms containing its square are negligible as compared to the terms containing its first power

- the change in volume is proportional to the initial volume V_0 and to the change in temperature according to the relationship

$$\Delta V = V_0\beta\Delta T,$$

where β is the **average coefficient of volume expansion**

Linear and volume expansion coefficients

- if expansion is isotropic — that is, the average coefficient of linear expansion of the object is the same in all directions —, linear and volume expansion coefficients are related through

  $\beta = 3\alpha$

- this can be proved by considering that the volume change can be expressed with the changes in the dimensions ℓ, w and h:

  $V_0 + \Delta V = (\ell_0 + \Delta \ell)(w_0 + \Delta w)(h_0 + \Delta h)$

  $V_0 + \Delta V = l_0(1 + \alpha \Delta T) \cdot w_0(1 + \alpha \Delta T) \cdot h_0(1 + \alpha \Delta T)$

  $V_0 + \Delta V = l_0 w_0 h_0 \cdot (1 + \alpha \Delta T)^3 = V_0 \cdot \left[1 + 3\alpha \Delta T + 3(\alpha \Delta T)^2 + (\alpha \Delta T)^3\right]$

- as α is usually very small, the terms containing its square and cube are negligible as compared to the terms containing its first power:

  $V_0 + \Delta V \approx V_0 + V_0 \cdot 3\alpha \Delta T$

  $\Delta V = V_0 \cdot 3\alpha \Delta T = V_0 \beta \Delta T$

  $\beta = 3\alpha$

Heat and internal energy

- *internal energy:* all the energy of a system that is associated with its microscopic components — atoms and molecules — when viewed from a reference frame at rest with respect to the centre of mass of the system
- any bulk kinetic energy of the system due to its motion through space is **not** included in internal energy (only the kinetic energy due to the random motion of its particles)
- internal energy includes kinetic energy of random translational, rotational, and vibrational motion of particles, potential energy within molecules, and potential energy between particles
- *heat:* the transfer of energy across the boundary of a system due to a temperature difference between the system and its surroundings
- the SI unit of heat is the joule:

$$[Q] = 1\,\mathrm{J}$$

Definition and state of an ideal gas

- *ideal gas:* a gas for which the following criteria are met:
    1. the only interaction between particles is through collisions (ie, there are no attractive forces)
    2. the volume of the particles of which the gas is made up is negligible compared to the volume of the container
- the state of the gas can be given by the following physical quantities:
    - the pressure p
    - the temperature T **(always expressed in Kelvins when using gas laws)**
    - the volume V
    - the mass m, the number of particles N or the amount of substance n

Quantities describing the amount of gas

- *Avogadro constant:* the number of atoms in exactly 12 grams of ^{12}C

$$N_A = 6.022 \cdot 10^{23} \, \frac{1}{\text{mol}}$$

- *amount of substance:* the number of particles present in a sample, as expressed as a multiple of the particles contained in exactly 12 grams of ^{12}C; the ratio of the number of particles in the sample to the Avogadro constant:

$$n = \frac{N}{N_A},$$

where N denotes the number of particles in the sample

- the SI unit of the amount of substance is the **mol**

- *molar mass:* the mass of one mole of a substance; the ratio of the mass m of the sample to the amount of substance n:

$$M = \frac{m}{n}$$

- molar mass is usually expressed in units of g/mol

Laws of the ideal gas

- **Boyle–Mariotte law:** at constant temperature, the pressure p and volume V of a fixed mass of gas are inversely proportional to each other:

    $pV = \text{constant}$

- **Gay-Lussac's first law:** at constant volume, the pressure p of a fixed mass of gas is proportional to the absolute temperature:

    $p \propto T$

- **Gay-Lussac's second law** or **Charles's law:** at constant pressure, the volume V of a fixed mass of gas is proportional to the absolute temperature:

    $V \propto T$

The equation of state for an ideal gas

- the Boyle–Mariotte law and the Gay-Lussac laws can be combined into a single **equation of state** which summarises all state changes:

  $$p\,V = n\,R\,T,$$

  where n is the amount of substance and R is the **universal gas constant**:

  $$R = 8.314 \, \frac{\text{J}}{\text{mol} \cdot \text{K}}$$

- an alternative form of the equation of state:

  $$p\,V = N\,k\,T,$$

  where N is the number of particles and k is **Boltzmann's constant:**

  $$k = \frac{R}{N_{\text{A}}} = 1.38 \cdot 10^{-23} \, \frac{\text{J}}{\text{K}}$$

- **in the equation of state (and in all gas laws), the temperature must always be expressed in kelvins**

Molecular model of an ideal gas

1. **The number of particles in the gas is large, and the average distance between them is large compared to the size of the particles** ⇒ particles take up a negligible volume in the container.

2. **The particles obey Newton's laws of motion, but as a whole they move randomly:** any particle can move in any direction with any speed.

3. **The particles only interact with each other through collisions.**

4. **The particles make elastic collisions with the walls.**

5. **All particles in the gas are identical.**

The container and the collisions with the walls
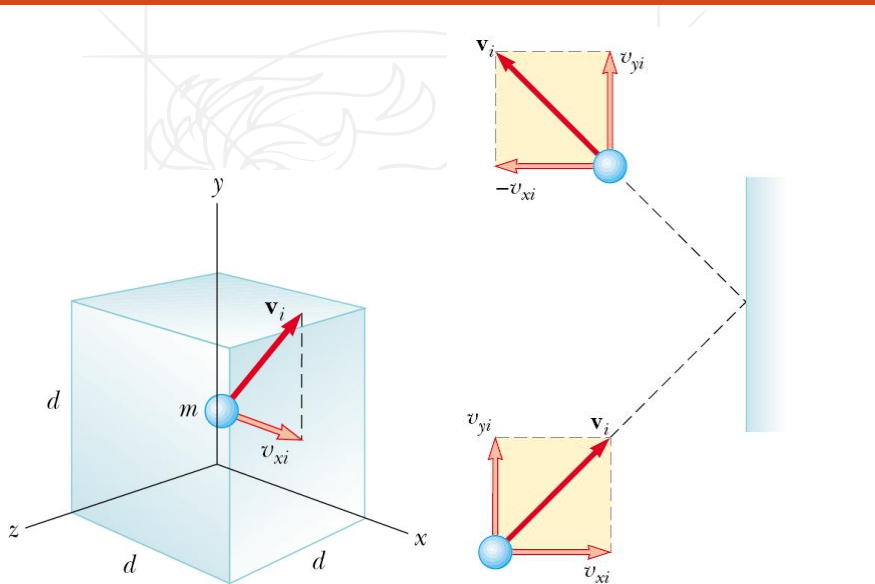
A collision in the x direction

- let us model the container as a cube with edges of d, and assume that all particles are identical with mass m
- assume that the ith particle makes an elastic collision with a wall perpendicular to the x direction; the velocity components after the collision are

$$v'_{yi} = v_{yi},$$

because the wall exerts no force in the y direction;

$$v'_{zi} = v_{zi},$$

because the wall exerts no force in the z direction; and

$$v'_{xi} = -v_{xi},$$

because the direction of motion changes, but the kinetic energy is conserved in an elastic collison, that is

$$v'^2 = v'^2_x + v'^2_y + v'^2_z = v^2 = v^2_x + v^2_y + v^2_z$$

The force acting on the particle

- the change in the x component of the momentum of the ith particle as a result of the collision is

  $$\Delta p_{xi} = -mv_{xi} - mv_{xi} = -2mv_{xi}$$

- according to the impulse-momentum theorem, this is equal to the impulse imparted to the particle by the wall in the duration of the collision:

  $$\Delta p_{xi} = F_{i,\text{ on particle}} \cdot \Delta t_{\text{collision}}$$

- we do not know how long the collision lasts, but we can rewrite the impulse-momentum theorem formula to contain the average force $\overline{F_i}$ acting on the particle in the time interval Δt that passes between two collisions with the same wall

  $$\Delta p_{xi} = \overline{F_i} \Delta t$$

- between two successive collisions, the particle must travel the length of the edge of the cube d twice, so

  $$\Delta t = \frac{2d}{v_{xi}}$$

The force acting on the wall

- this way, the average force exerted by the wall on the ith particle is

$$\overline{F_i} = \frac{\Delta p_{xi}}{\Delta t} = -2m v_{xi} \cdot \frac{v_{xi}}{2d} = -\frac{m}{d} \cdot v_{xi}^2$$

- Newton's third law states that the force exerted on the wall by the ith particle is the the opposite of $\overline{F_i}$:

$$\overline{F_{i,\ \text{wall}}} = -\overline{F_i} = \frac{m}{d} \cdot v_{xi}^2$$

- this is the force exerted only by the ith particle; to get the total force $\overline{F}$ acting on the wall, we have to add up the forces from all the N particles:

$$\overline{F} = \sum_{i=0}^{N-1} \frac{m}{d} \cdot v_{xi}^2 = \frac{m}{d} \sum_{i=0}^{N-1} v_{xi}^2$$

- if there were only a few particles, the average force would fluctuate in time, but for a large number of particles, the variations are smoothed out, so the average force will be the same for any time interval
- **the force on the wall, F, is constant in time**

The average speed of particles

- particles have different velocities and speeds, but we can take an average of them
- the average speed square in the x direction is

$$\overline{v_x^2} = \frac{1}{N} \sum_{i=0}^{N-1} v_{xi}^2$$

- thus the sum in the expression of the force is

$$\sum_{i=0}^{N-1} v_{xi}^2 = N \overline{v_x^2}$$

- the force exerted on the wall becomes

$$F = N \frac{m}{d} \overline{v_x^2}$$

- the pressure on a wall is the force divided by the surface area of the wall

$$p = \frac{F}{A} = \frac{F}{d^2} = N \frac{m}{d^3} \overline{v_x^2} = N \frac{m}{V} \overline{v_x^2},$$

where $V = d^3$ is the volume of the container (remember, it is a cube)

The equipartition theorem

- equipartition: from Latin, means 'equal sharing'
- in thermal equilibrium, each particle has an equal share in the energy
- *theorem of equipartition of energy:* in thermal equilibrium, each particle in the system has an equal average energy in each degree of freedom:

$$\overline{\epsilon} = \frac{1}{2} k T$$

 where $\overline{\epsilon}$ is the average energy in a degree of freedom, k is Boltzmann's constant and T is the absolute temperature
- *degree of freedom:* an independent means by which a molecule can possess energy
- the number of degrees of freedom (f):
  - translation: 3 degrees of freedom, corresponding to the 3 dimensions of space
  - rotation: no extra degrees of freedom for monatomic gases, 2 degrees of freedom for diatomic molecules and 3 for polyatomic molecules
  - vibrational: degrees of freedom associated with the vibration around equilibrium positions in molecules; depend on structure

Degrees of freedom for a diatomic molecule
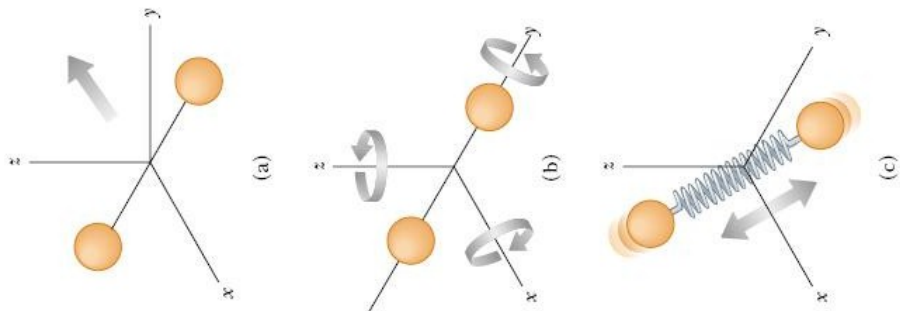
Degrees of freedom for different gases

Monatomic

3 translational degrees of freedom: $f = 3$

Diatomic

- 3 translational
- 2 rotational (the contribution from rotating around the longitudinal axis is negligible as compared to the other two axes)
- 2 vibrational: kinetic and potential energy components of vibrations
- quantum mechanics $\Rightarrow$ at low temperatures, all molecules are in the lowest possible rotational and vibrational energy states, so those degrees of freedom are 'frozen'; at room temperature, the rotational components begin to contribute but the vibrational ones still do not; only at high temperatures are the vibrational degrees of freedom taken into account
- so at room temperature the number of the degrees of freedom is $f = 3 + 2 = 5$

Polyatomic

for similar reasons as with diatomic gases, we can assume that the number of the degrees of freedom at room temperature is $f = 3 + 3 = 6$ (3 translational, 3 rotational)

The equation of state of the ideal gas

- we have found that the pressure on a wall is

$$p = N \frac{m}{V} \overline{v_x^2},$$

- from the equipartition theorem, we know that the average energy associated with translational motion in the x direction is

$$\overline{\epsilon_x} = \frac{1}{2} k T$$

- this average energy is the kinetic energy of the motion in the x direction, so

$$\overline{\epsilon_x} = \frac{1}{2} m \overline{v_x^2}$$

- comparing the two equations, we get that

$$m \overline{v_x^2} = k T$$

- this result tells us that temperature is a direct measure of the average kinetic energy of the particles

- substituting it into the formula of the pressure, and multiplying both sides by V, we get the equation of state of the ideal gas:

$$p V = N m \overline{v_x^2} = N k T$$

Laws of thermodynamics

0. If objects A and B are separately in thermal equilibrium with a third object C, then A and B are in thermal equilibrium with each other.

1. The change in the internal energy of a system is equal to the sum of the heat Q transferred **to** the system and the work W done **on** the system.

2. Forms:
   - It is impossible to construct a heat engine that, operating in a cycle, produces no effect other than the input of energy by heat from a reservoir and the performance of an equal amount of work.
   - Energy does not transfer spontaneously by heat from a cold object to a hot object.
   - The total entropy of an isolated system that undergoes a change cannot decrease.

3. The entropy of chemically homogeneous substances approaches a constant value when the temperature approaches $0\,\text{K}$.

The first law of thermodynamics

- conservation of energy: the change in the energy of the system is equal to the total amount of energy transferred through the boundary of the system
- the first law of thermodynamics: a special case of the conservation of energy
- ways to change the internal energy of a system
    - **work done on the system:** an external force acts on the system which causes a macroscopic displacement
    - **heat:** a microscopic form of energy transfer if a temperature difference acts across the boundary of the system
- *the first law of thermodynamics:* the change in the internal energy of a system is equal to the sum of the heat Q transferred **to** the system and the work W done **on** the system:

$$\Delta E_{\text{int}} = Q + W$$

- sign conventions:
    - Q is positive when the system gains heat and negative when loses
    - W is positive when the system gains energy through work (eg, compression of a gas) and negative when it loses energy through work (eg, expansion of a gas)

Work done on a gas
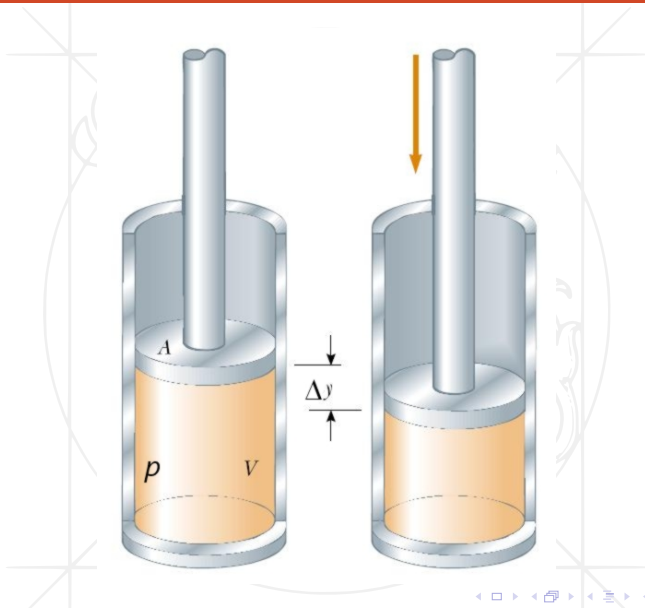
Work done on a gas

- assume that we have a gas in a cylinder closed with a piston; the pressure of the gas is p and the surface area of the piston is A, so the force exerted by the gas on the piston is pA
- according to Newton's third law, the force exerted on the gas is the opposite, $-pA$
- on a small displacement Δy, this force does ΔW work on the gas:

    $\Delta W = -pA\Delta y = -p\Delta V,$

    as $A\Delta y$ is the volume change the gas is subjected to
- **if the pressure stays constant** during the process, we can express the work with a simple formula:

    $W = -p\Delta V,$

    where $\Delta V = V_\mathrm{f} - V_\mathrm{i}$ is the total volume change during the process
- generally, the pressure varies with the volume ($p = p(V)$), so we have to use integration:

$$W = -\int_{V_\mathrm{i}}^{V_\mathrm{f}} p(V)\,\mathrm{d}V$$

Work done on a gas

- the simplified meaning of integration: 'the area under the $p(V)$ curve'

- the volume change:
    - positive when the gas expands
    - negative when the gas is compressed

- thus **the work done on the gas** is
    - **negative when the gas expands**
    - **positive when the gas is compressed**

Internal energy of a gas

- internal energy: the energy associated with the microscopic components of a system
- in an ideal gas, there are no interactions between particles except for collisions, so the only source of energy is kinetic energy
- according to the equipartition theorem, the average energy per particle per degree of freedom is

  $$\overline{\epsilon} = \frac{1}{2} k T$$

- so, for a gas containing N number of particles, each with f degrees of freedom, the total energy, which is the internal energy, is

  $$E_{\text{int}} = \frac{f}{2} N k T$$

- the degrees of freedom at room temperature: $f = 3$ for monatomic gases, $f = 5$ for diatomic gases and $f = 6$ for polyatomic gases
- using the equation of state $p V = N k T$, we can express the internal energy in another form:

  $$E_{\text{int}} = \frac{f}{2} p V$$

Special thermodynamic processes (1)

Adiabatic process

- **no heat** enters or leaves the system: $\boxed{Q = 0}$
- $\Delta E_{\text{int}} = W$ (for an adiabatic process)

Isobaric process

- a process that occurs at **constant pressure**
- at constant pressure, the work done on the gas is simplified to $W = -p\,(V_{\text{f}} - V_{\text{i}})$

Special thermodynamic processes (2)

Isovolumetric process

- a process that occurs at **constant volume**
- at constant volume, the **work done on the gas is zero** as the volume does not change
- $\Delta E_{\text{int}} = Q$ (for an isovolumetric process)

Isothermal process

- a process that occurs at **constant temperature**
- at constant temperature, the **change in the internal energy is zero:**

$$\Delta E_{\text{int}} = \frac{f}{2}NkT - \frac{f}{2}NkT = 0$$

- $Q = -W$ (for an isothermal process)

Real gases

- real gases:
  - not particles (non-zero volume)
  - attractive forces between particles
- van der Waals equation of state

$$\left(p + \frac{n^2 a}{V^2} \right)(V - nb) = nRT,$$

- where a is a measure of the attraction between the particles and b is the volume excluded by a mole of particles.

Pressure-depth relation
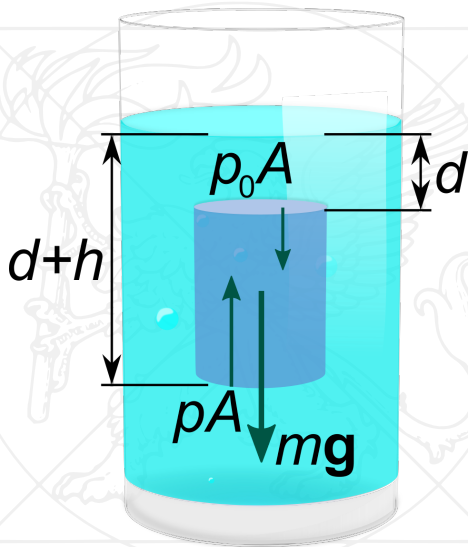
Pressure-depth relation

- an incompressible liquid of density ϱ at rest
- we pick the liquid contained within an imaginary cylinder of cross-sectional area A extending from depth d to depth $d + h$
- the external liquid exerts forces at all points on the surface of the sample, perpendicular to the surface
- pressure on top face: $p_0 \Rightarrow$ force: $F_0 = p_0 A$
- pressure on bottom face: $p \Rightarrow$ force: $F = pA$
- the liquid volume is at rest, so the net force acting on it must be zero
- equilibrium condition in the vertical direction:

$$\sum F_y = 0 = pA - mg - p_0 A$$

- the mass of the sample can be expressed using the density and the volume:

$$m = \varrho V = \varrho A h$$

Pressure-depth relation and Pascal's law

- substituting this into the equilibrium equation

$$0 = pA - \varrho A h g - p_0 A$$

- dividing both sides by A and rearranging, we get

$$p = p_0 + \varrho h g$$

- it means that the pressure at a depth h below a given point is greater by an amount $\varrho h g$
- pressure below the surface depends the pressure on the surface and on the depth below the surface
- if the pressure on the surface changes, the pressure below must also change $\Rightarrow$ any increase in pressure at the surface must be transmitted to every other point in the fluid
- *Pascal's law:* a change in the pressure applied to a stationary fluid is transmitted undiminished to every point of the fluid and to the walls of the container.

Hydrostatic paradox

It follows from Pascal's law that the pressure acting on the bottom of a container is independent of the shape of the container; it only depends on the height of the fluid. The pressure on the bottom surface is the same for all the containers below.

Barometer
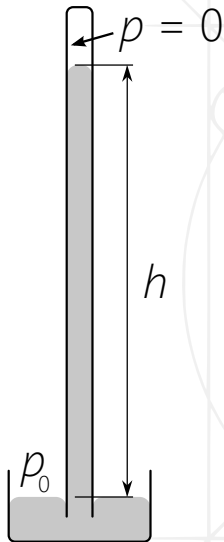
- invented by EVANGELISTA TORRICELLI (1608−1647)
- a tube closed at one end is filled with mercury, then inverted into a dish of mercury
- the pressure inside the tube (the hydrostatic pressure of mercury, $p = \varrho_{Hg}hg$) keeps equilibrium with the pressure outside (atmospheric pressure, p_0):

$$p_0 = \varrho_{Hg}hg$$
$$h = \frac{p_0}{\varrho_{Hg}g} = \frac{1.013 \cdot 10^5 \, \text{Pa}}{\left(13600 \, \text{kg/m}^3\right) \cdot \left(9.8 \, \text{m/s}^2\right)} = 0.76 \, \text{m}$$

- as the atmospheric pressure changes, the height also changes ⇒ the tube can be calibrated and scaled to measure atmospheric pressure
- the most common pressure unit in medicine is millimetre of mercury (mmHg): $1 \, \text{mmHg} = \left(13600 \, \text{kg/m}^3\right) \cdot \left(9.8 \, \text{m/s}^2\right) \cdot \left(0.001 \, \text{m}\right) = 133.3 \, \text{Pa}$

Hydrostatics in the human body
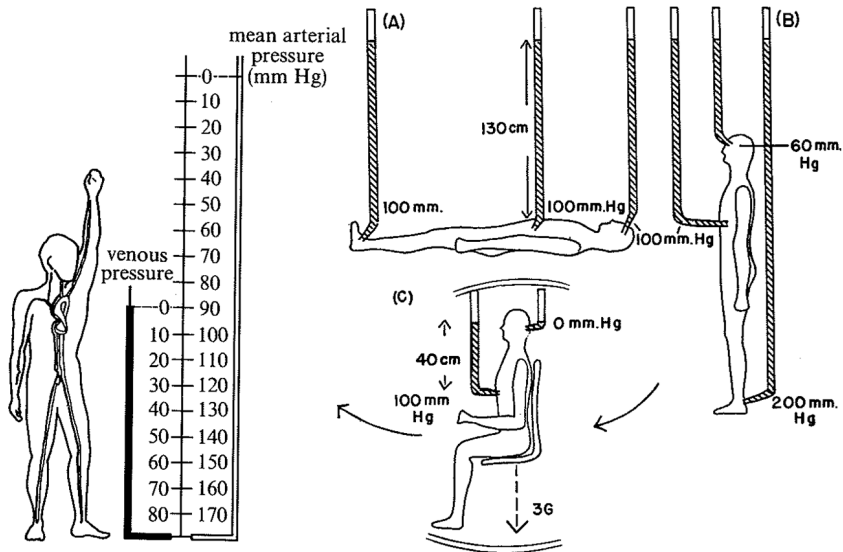
Basic quantities

- *volumetric flow rate* or *volume current* the volume flowing through a cross-section of a tube in unit time

$$I := \frac{\mathrm{d}V}{\mathrm{d}t},$$

  where m denotes the mass, t is the time and V stands for the volume

- the greater the cross-section area, the greater the volumetric flow rate $\rightarrow$
- *volumetric flux* (or *volumetric current density*): the volumetric flow rate through a unit cross-section area

$$J := \frac{\mathrm{d}I}{\mathrm{d}A},$$

  where A denotes the cross-section area

What drives fluid flow?

- analogue: Ohm's law — when there is a potential difference, electrical current will flow in proportion to the potential difference (voltage): $I = \frac{1}{R}U$
- in fluid flow, pressure will take the place of the electrical potential and volumetric flow rate will take the place of current — when there is a pressure difference, the fluid will flow in proportion, from high-pressure areas towards low-pressure areas
- further analogues:
  - diffusion: when there is a concentration difference, molecules will diffuse in proportion, from high-concentration areas towards low-concentration areas
  - heat flow: when there is a temperature difference, heat will flow in proportion, from hotter areas towards colder areas
  - falling objects: when there is difference in gravitational potential energy, objects will fall (or roll down) from high-potential areas towards low-potential areas
- this is a general pattern in Nature: when there is a difference in a potential (electric, gravitational, pressure, temperature, concentration), a transport process will occur, directed from the high-potential area towards the low-potential area

Flow of incompressible fluids

- liquids: generally incompressible
- gases: generally compressible
- even for gases, if the flow speed is not too high ($< 50\,\text{m/s}$), ϱ does not change significantly and the *flow* can be considered incompressible
- time dependence of the flow
  - **stationary (steady):** speed and flow rate are *independent of* time
  - **non-stationary (unsteady):** speed and flow rate *do depend on* time

Flow rate and flow speed
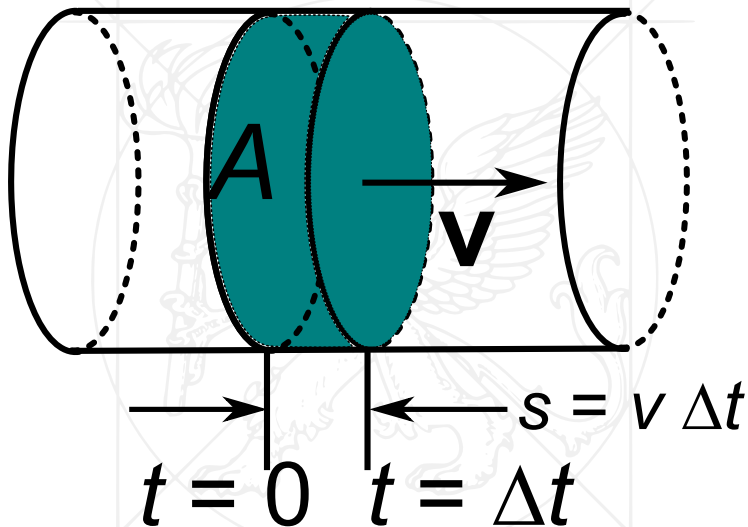
Flow rate and flow speed

- let us assume that the fluid particles move parallel to each other with the same flow velocity **v**
- this way, in time Δt they travel a distance $s = v\Delta t$
- from the perspective of fluid flow, this means that volume ΔV has been carried through a given cross-section area A of the tube, where

$$\Delta V = As = Av\Delta t$$

- thus the volumetric flow rate of the flow is

$$I = \frac{\Delta V}{\Delta t} = \frac{Av\Delta t}{\Delta t} = Av$$

Illustration
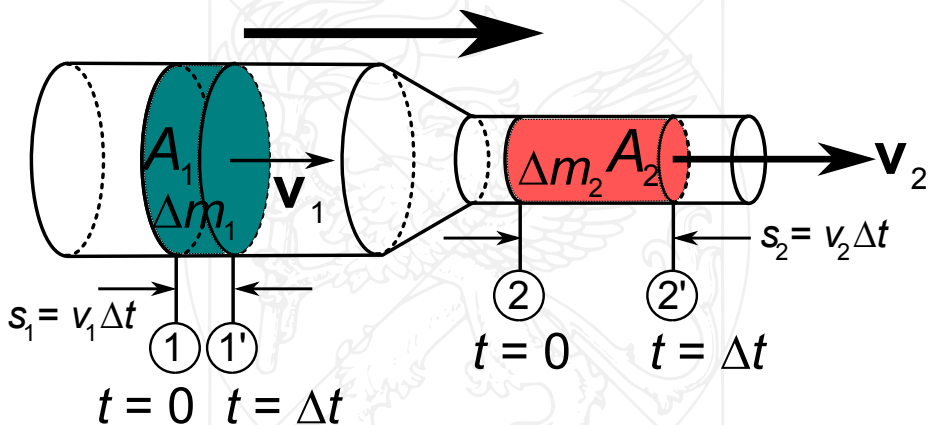
Notations

- (1) and (2): selected discs in the fluid at time $t = 0$
- (1') and (2'): locations of the selected discs in at time $t = \Delta t$
- $\mathbf{v}_1$: the velocity of the fluid at (1) and (1')
- $\mathbf{v}_2$: the velocity of the fluid at (2) and (2')
- A_1: area of the cross-section at (1) and (1')
- A_2: area of the cross-section at (2) and (2')
- s_1: distance travelled by the fluid between locations (1) and (1')
- s_2: distance travelled by the fluid between locations (2) and (2')

Consequences of incompressibility

- the density is the same between (1) and (1') and (2) and (2'): $\varrho_1 = \varrho_2 = \varrho$
- the mass flowing in at (1) is equal to the mass flowing out at (2): $\Delta m_1 = \Delta m_2 = \Delta m$
- otherwise mass would accumulate or run out in the volume of the pipe between (1) and (2), increasing or decreasing the density, which contradicts the condition of incompressibility
- $\Delta m = \varrho \Delta V_1 = \varrho \Delta V_2 \rightarrow$
- the volume flowing in at (1) is equal to the volume flowing out at (2): $\Delta V_1 = \Delta V_2 = \Delta V$

Equation of continuity

- we have seen that since the fluid is incompressible,

    $$\Delta V_1 = \Delta V_2$$

- $\Delta V_1 = A_1 s_1 = A_1 v_1 \Delta t$
- $\Delta V_2 = A_2 s_2 = A_2 v_2 \Delta t$
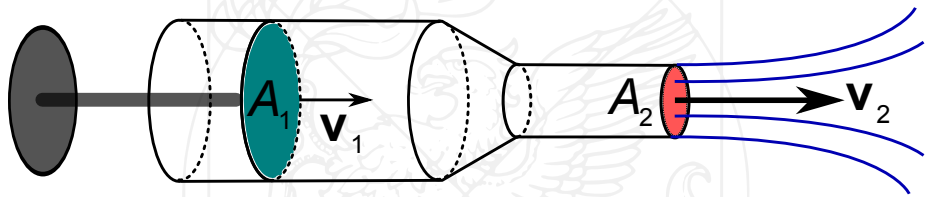- $\Delta V_1 = \Delta V_2 \Rightarrow A_1 v_1 \Delta t = A_2 v_2 \Delta t$

    $$A_1 v_1 = A_2 v_2$$

- since (1) and (2) were chosen arbitrarily, this must hold to any two cross-sections along the flow:

    $$A \cdot v = \text{const}$$

- this is the *equation of continuity*
- since $Av = I$, this means that the **volumetric flow rate is constant along the tube**

Example: a syringe

$$A_1 v_1 = A_2 v_2$$

$$A_1 \gg A_2 \Rightarrow v_2 \gg v_1$$

Example: flow speed in blood vessels

Blood vessel	cross-section area [cm^2]	velocity [cm/s]
Aorta	4.5	40
Arteries	20	9
Arterioles	400	0.45
Capillaries	4500	0.04
Veins	40	4.5
Vena cava	18	10

Example: flow speed in blood vessels
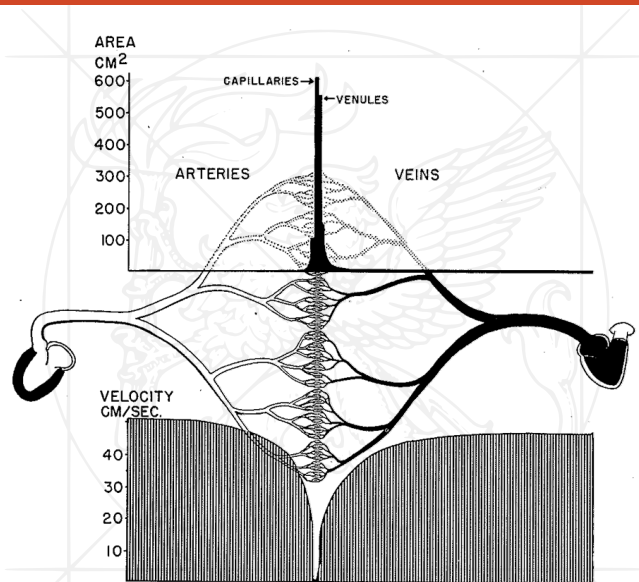
Flow of ideal fluids

- ideal fluid: incompressible and **non-viscous**
- no internal friction $\rightarrow$ no loss of mechanical energy
- mechanical energy: potential energy + kinetic energy

$$E = E_{\mathrm{P}} + E_{\mathrm{K}}$$

- the *conservation of mechanical energy* applies: external work done on the system = change in the mechanical energy of the system

$$W = \Delta E = \Delta E_{\mathrm{P}} + \Delta E_{\mathrm{K}}$$

- Bernoulli's law: a special form of the conservation of mechanical energy for the flow of ideal fluids

Illustration
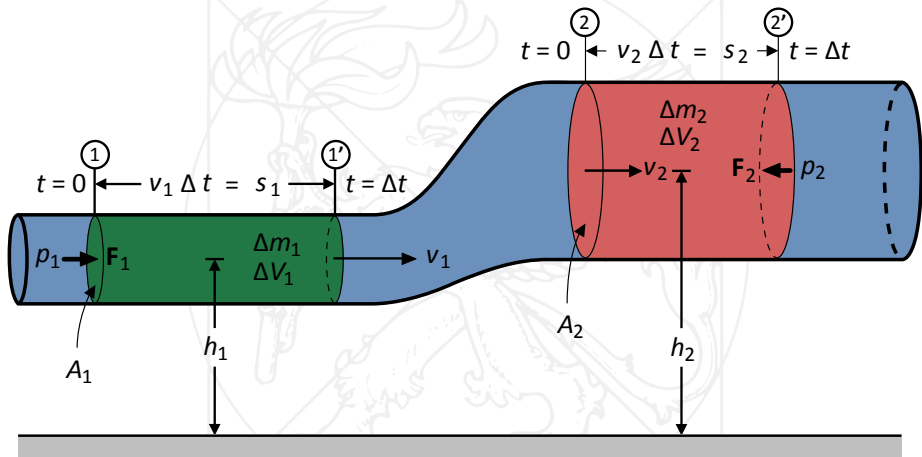
Notations

- (1) and (2): selected discs in the fluid at time $t = 0$
- (1') and (2'): locations of the selected discs in at time $t = \Delta t$
- $\mathbf{v}_i$: the velocity of the fluid at (i) and (i') ($i \in \{1, 2\}$)
- A_i: area of the cross-section at (i) and (i') ($i \in \{1, 2\}$)
- s_i: distance travelled by the fluid between locations (i) and (i') ($i \in \{1, 2\}$)
- h_i: height of the centre of mass of the fluid at (i) and (i') (as compared to an arbitrary reference level, $i \in \{1, 2\}$)
- p_i: pressure exerted by the rest of the fluid at (i) and (i') ($i \in \{1, 2\}$)
- $\mathbf{F}_i$: force exerted by the rest of the fluid at (i) and (i') ($i \in \{1, 2\}$)
- Δm_i: mass of the fluid between locations (i) and (i') ($i \in \{1, 2\}$)
- ΔV_i: volume of the fluid between locations (i) and (i') ($i \in \{1, 2\}$)
- ϱ_i: density of the fluid between locations (i) and (i') ($i \in \{1, 2\}$)

Energy changes

- how does the energy of the fluid between (1) and (2) change?
- kinetic and potential energies only depend on the position
- E_C: the energy of the part of the fluid which does not change between $t = 0$ and $t = \Delta t$
- $E = E_1 + E_C$
- $E' = E_C + E_2$
- $\Delta E = E' - E = E_2 - E_1$
- only the energy of the marked sections changes
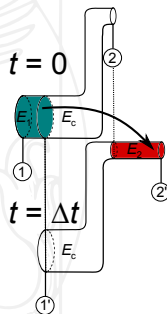
Implications of the equation of continuity

$$\varrho_1 = \varrho_2 = \varrho$$

$$\Delta m_1 = \Delta m_2 = \Delta m$$

$$\Delta V_1 = \frac{\Delta m_1}{\varrho} = \frac{\Delta m_2}{\varrho} = \Delta V_2$$

$$\Delta V_1 = A_1 s_1 = A_1 v_1 \Delta t$$

$$\Delta V_2 = A_1 s_1 = A_2 v_2 \Delta t$$

Difference between the fluid states at $t = 0$ and $t = \Delta t$: mass Δm is transported from (1) to (2)

What happens to this mass?

1. the rest of the fluid does work on it
2. moved from h_1 to $h_2 \rightarrow$ its potential energy changes
3. its velocity changes $\rightarrow$ its kinetic energy changes

Work done by the rest of the fluid

- definition of pressure:

$$p = \frac{F}{A}$$

$$F = pA$$

- work done by the fluid at (1) and (2):

$$W_1 = F_1 s_1 = F_1 v_1 \Delta t = p_1 A_1 v_1 \Delta t = p_1 \Delta V = p_1 \frac{\Delta m}{\varrho}$$

$$W_2 = -F_2 s_2 = -F_2 v_2 \Delta t = -p_2 A_2 v_2 \Delta t = -p_2 \Delta V = -p_2 \frac{\Delta m}{\varrho}$$

- negative sign in W_2: the direction of the force is opposite to that of the displacement

- total external work:

$$W = W_1 + W_2 = \frac{\Delta m}{\varrho}(p_1 - p_2)$$

Potential and kinetic energies

Gravitational potential energy: $E_P = m\,g\,h$

- at (1): $E_{P,1} = \Delta m\,g\,h_1$
- at (2): $E_{P,2} = \Delta m\,g\,h_2$
- change: $\Delta E_P = E_{P,2} - E_{P,1} = \Delta m g (h_2 - h_1)$

Kinetic energy: $E_K = \frac{1}{2} m\,v^2$

- at (1): $E_{K,1} = \frac{1}{2} \Delta m\,v_1^2$
- at (2): $E_{K,2} = \frac{1}{2} \Delta m\,v_2^2$
- change: $\Delta E_K = E_{K,2} - E_{K,1} = \frac{1}{2} \Delta m (v_2^2 - v_1^2)$

Bernoulli's equation

- conservation of mechanical energy: $W = \Delta E = \Delta E_P + \Delta E_K$

$$\frac{\Delta m}{\varrho}(p_1 - p_2) = \Delta m g(h_2 - h_1) + \frac{1}{2}\Delta m(v_2^2 - v_1^2) \qquad // \cdot \frac{\varrho}{\Delta m}$$

$$p_1 - p_2 = \varrho g h_2 - \varrho g h_1 + \frac{1}{2}\varrho v_2^2 - \frac{1}{2}\varrho v_1^2$$

$$p_1 + \varrho g h_1 + \frac{1}{2}\varrho v_1^2 = p_2 + \varrho g h_2 + \frac{1}{2}\varrho v_2^2$$

- fluid discs (1) and (2) were chosen arbitrarily, so this must hold to any two cross-sections along the flow: $\quad p + \varrho g h + \frac{1}{2}\varrho v^2 = \text{const}$

- this is **Bernoulli's law**

Interpreting Bernoulli's law

- p: ***static pressure***
- $\varrho\,g\,h$: ***hydrostatic pressure*** — gravitational potential energy per unit volume
- $\frac{1}{2}\varrho\,v^2$: ***dynamic pressure*** — kinetic energy per unit volume
- ***total pressure*** = static pressure + hydrostatic pressure + dynamic pressure
- Bernoulli's law in other words: *the total pressure is constant along the tube*

# Applying Bernoulli's law: pressure–depth relationship
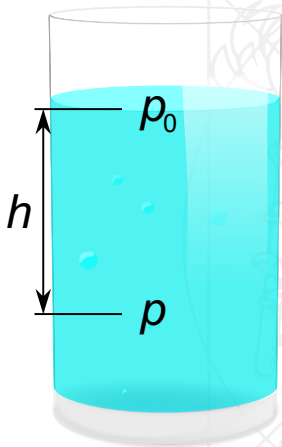
- we compare two points in a stationary fluid, one located deeper than the other by h
- the dynamic pressure is zero at both locations
- the static pressure is p_0 in the higher point, p in the lower point
- let the reference level of the hydrostatic pressure be at the lower point $\Rightarrow$ the hydrostatic pressure is zero in the lower point and $\varrho\,h\,g$ in the higher point
- applying Bernoulli's law:

$$p_0 + 0 + \varrho\,h\,g = p + 0 + 0$$

$$p_0 + \varrho\,h\,g = p$$

- this is the pressure–depth relationship we obtained earlier

Example: Venturi tube
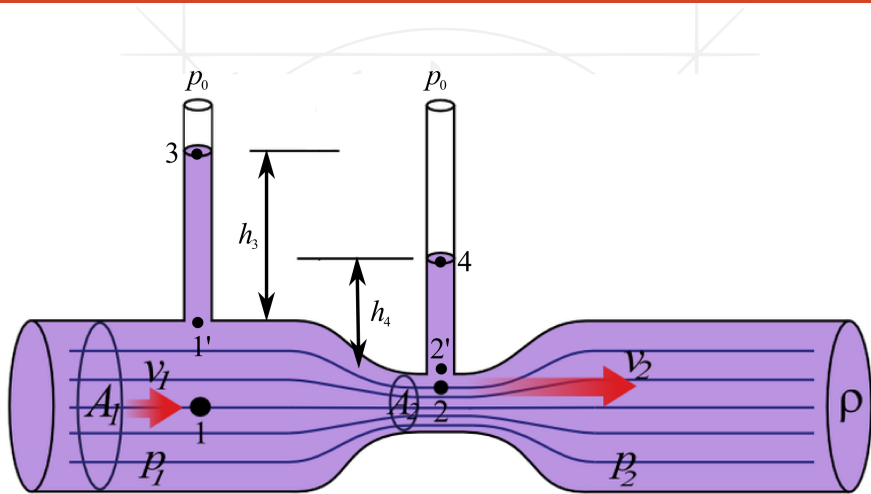
Example: Venturi tube

- from the equation of continuity:

  $$A_1 v_1 = A_2 v_2$$

  $$A_1 \gg A_2 \Rightarrow v_2 \gg v_1$$

- apply Bernoulli's law to compare 1 and 2:

  $$p_1 + 0 + \frac{1}{2}\varrho\, v_1^2 = p_2 + 0 + \frac{1}{2}\varrho\, v_2^2$$

- since $v_2 \gg v_1 \Rightarrow p_1 \gg p_2$
- 1 and 3 cannot be compared, because they are in different tubes; but 1' and 3 can (1' is at the beginning of the vertical tube)
- the static pressures at 1 and 1' are the same: $p_{1'} = p_1$
- the dynamic pressure at 1' and 3 is 0, because the fluid does not flow in the vertical tubes
- apply Bernoulli's law to compare 1' and 3 (the reference level is now at 1'):

  $$p_1 + 0 + 0 = p_0 + \varrho\, h_3 g + 0,$$

  where p_0 is the atmospheric pressure

Example: Venturi tube

- 2 and 4 cannot be compared, because they are in different tubes; but 2' and 4 can (2' is at the beginning of the vertical tube)
- the static pressures at 2 and 2' are the same: $p_{2'} = p_2$
- the dynamic pressure at 2' and 4 is 0, because the fluid does not flow in the vertical tubes
- apply Bernoulli's law to compare 2' and 4 (the reference level is now at 2'):

$$p_2 + 0 + 0 = p_0 + \varrho\, h_4 g + 0$$

- if we compare h_3 and h_4

$$h_3 = \frac{p_1 - p_0}{\varrho\, g} \gg h_4 = \frac{p_2 - p_0}{\varrho\, g},$$

  because we have seen that $p_1 \gg p_2$

- the Venturi tube proves Bernoulli's law — *at the wider section of the tube, where flow speed is smaller, the static pressure is higher as compared to narrower sections of the tube*
- the role of vertical tubes: to make the differences in static pressure visible
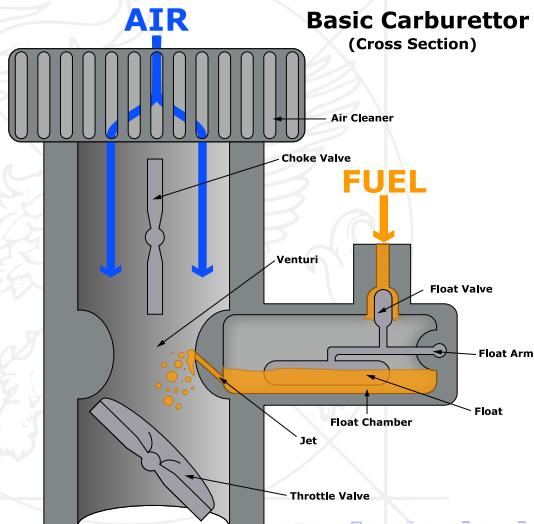
Example: Bunsen burner

- flow speed in the narrower section is greater $\Rightarrow$ lower static pressure
- static pressure within the tube is less than the atmospheric pressure
- as a result, air flows into the tube
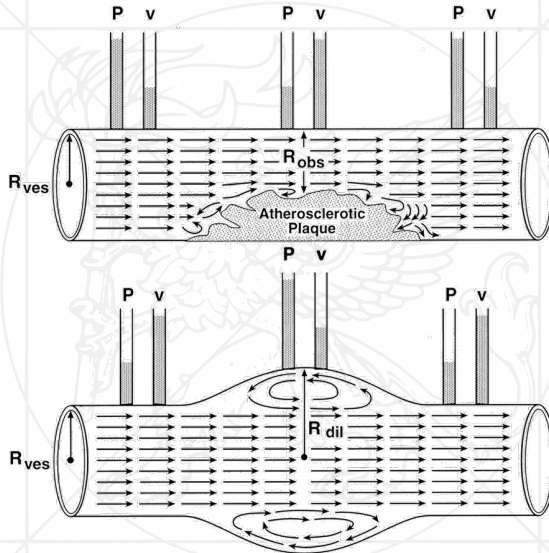- the air influx feeds the flame at the top
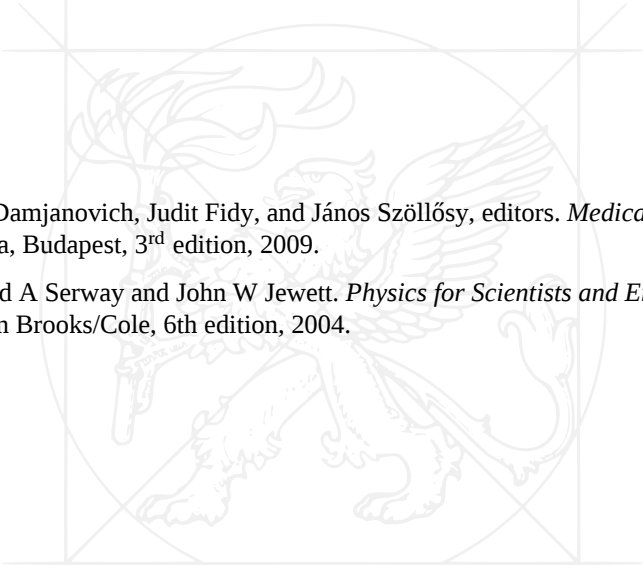
Example: carburettor

- air injected into the carburettor chamber

- in the narrow throat, the air is moving at its fastest speed and therefore it is at its lowest pressure

- low pressure in the chamber pumps the fuel into the chamber, where it is mixed with air

AIR

Basic Carburettor
(Cross Section)

Air Cleaner

Choke Valve

FUEL

Venturi

Float Valve

Float Arm

Float

Float Chamber

Jet

Throttle Valve

Examples: atherosclerosis and aneurysm

Sources and recommended reading I

[1] Sándor Damjanovich, Judit Fidy, and János Szöllősy, editors. *Medical biophysics*. Medicina, Budapest, 3rd edition, 2009.

[2] Raymond A Serway and John W Jewett. *Physics for Scientists and Engineers*. Thomson Brooks/Cole, 6th edition, 2004.