Mechanical waves, acoustics and ultrasounds

Lecture for freshmen medical students

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Preparation for the exam

Lecture (slide show, see the home page of the Institute)
Handout (more text, see the home page of the Institute)
Seminars

Be careful! Not all fields are expressed explicitely in the lecture or handout. Some topics will be dealt in implicite forms of problems. Try to solve them either at home and/or at seminars.

Suggested texts to consult

J. J. Braun: Study Guide: Physics for Scientists and Engineers, HarperCollinsCollegePublishers, New York 1995 or any other college physics texts.

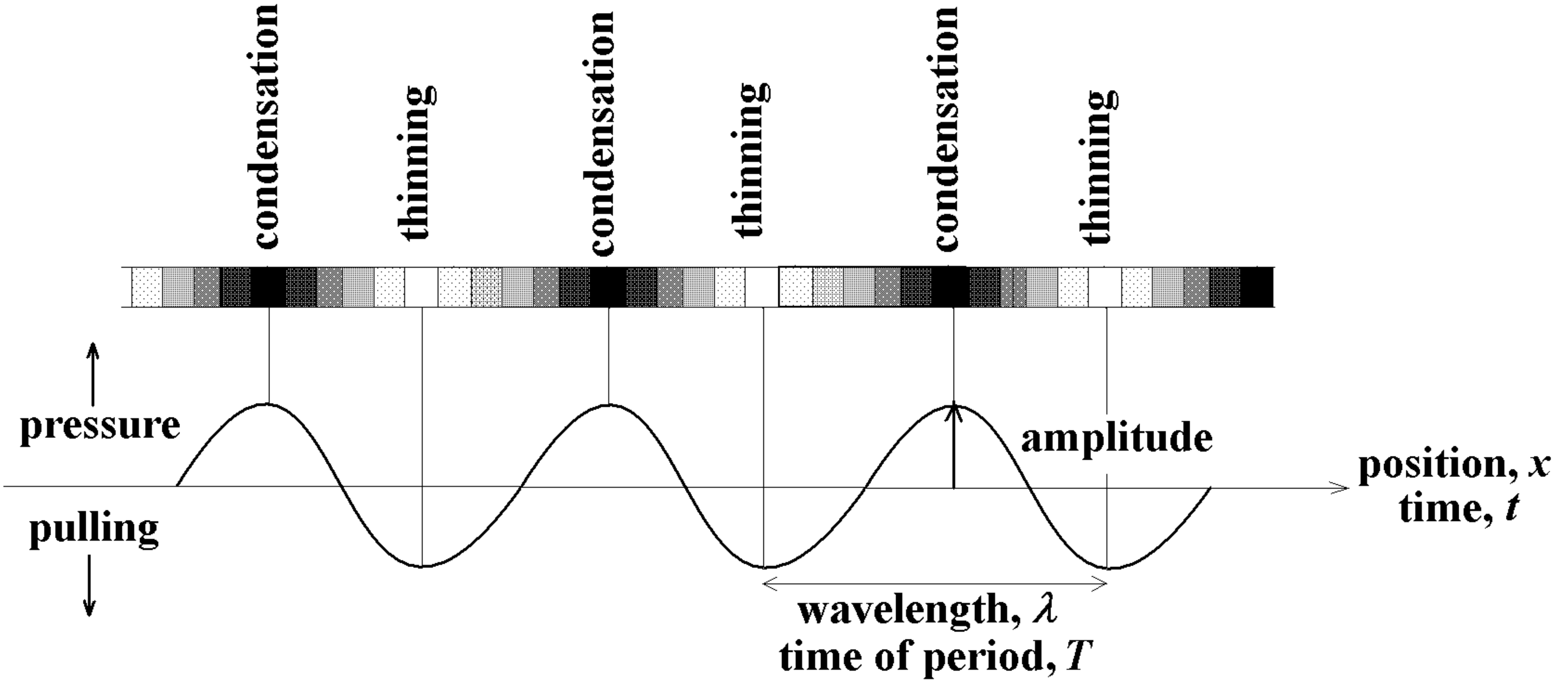
P. Maróti, I. Berkes and F. Tölgyesi: Biophysics Problems. A textbook with answers, Akadémiai Kiadó, Budapest 1998.

S. Damjanovich, J. Fidy and J. Szöllősi (eds.): Medical Biophysics, Medicina, Budapest, 2009.

Mechanical waves

The energy of an oscillating particle in elastic medium can propagate both in space and in time.

Propagation of the wave along straight line; transverse and longitudinal waves



Spectrum of mechanical waves (sound)

ultrasound

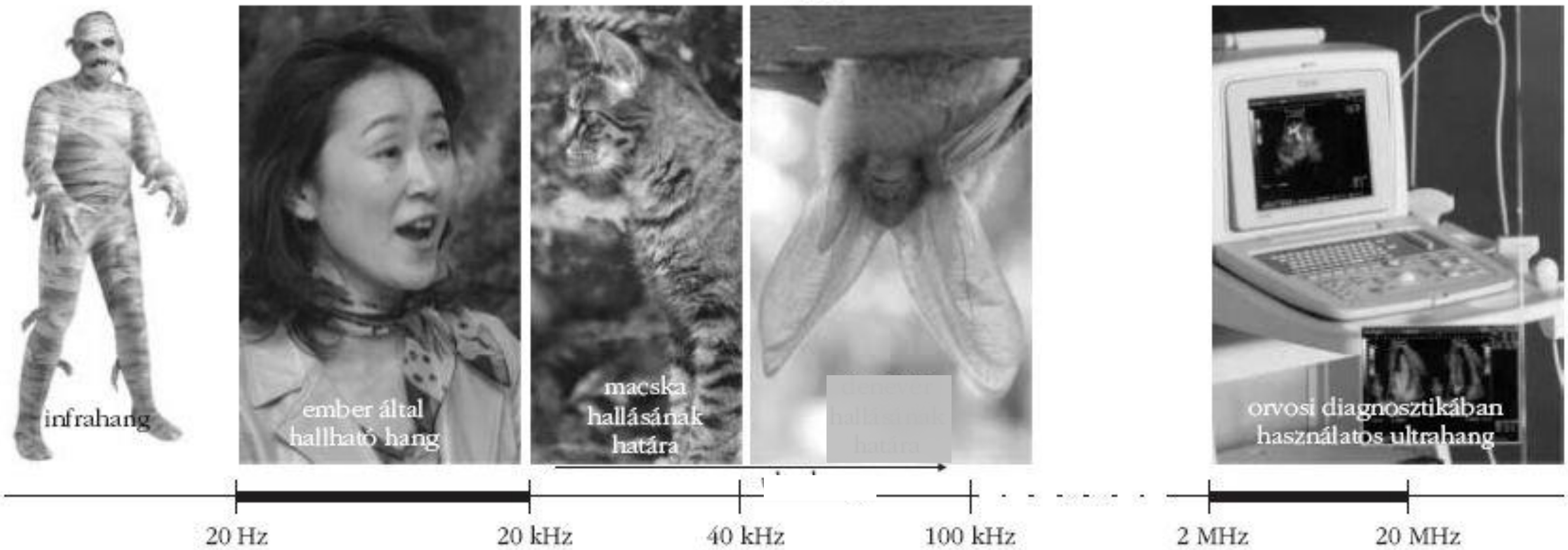
infrasounds

human speech,
human hearing

upper limit of
the audibility of
the cat

bat

ultrasound for
medical
application



Equation of the one dimensional harmonic wave

$$y(x=0,t) = A\sin(\omega t + \varphi)$$

$$y(x,t) = A\sin\left(\omega(t - x/c) + \varphi\right)$$

Wave number: $\bar{\lambda} = \dfrac{2\pi}{\lambda}$

Phase angle: $\left(\omega(t - x/c) + \varphi\right)$

Speed of propagation: *c*

Linear frequency: *f*

$$\lambda = c \cdot T = \frac{c}{f} = 2\pi\frac{c}{\omega}$$

Angular frequency: *ω*

Wave number

Initial phase: φ

$$y(x,t) = A\sin\left(\omega t - \bar{\lambda}x + \varphi\right)$$

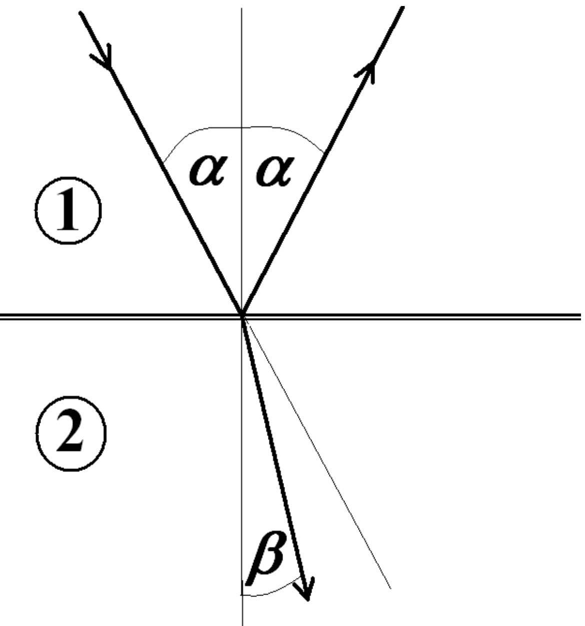
Time of period: T

Wavelength: (λ)

Initial phase: φ

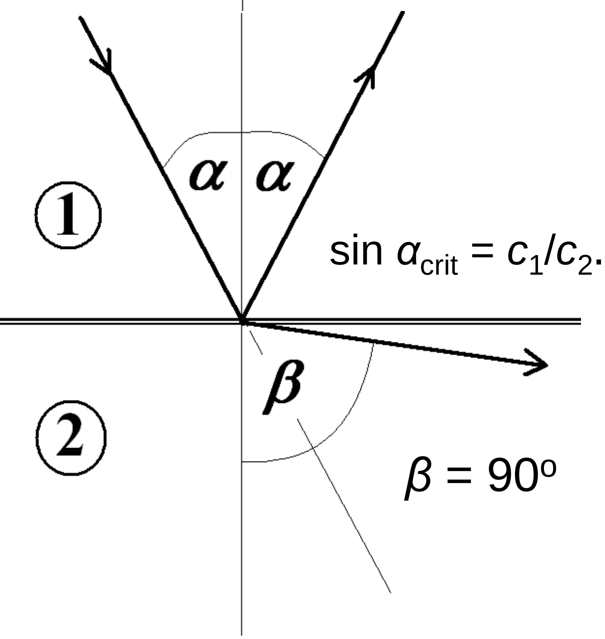
$$y(x,t) = A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda} + \varphi/2\pi\right)$$

Reflection and refraction of sound



$$\alpha = \alpha_{refl} \qquad \frac{\sin\alpha}{\sin\beta} = \frac{c_1}{c_2} = n_{21} = \text{const.}$$

The crtical angle of total reflection (β = 90^o):
sin $\alpha_{crit} = c_1/c_2$.

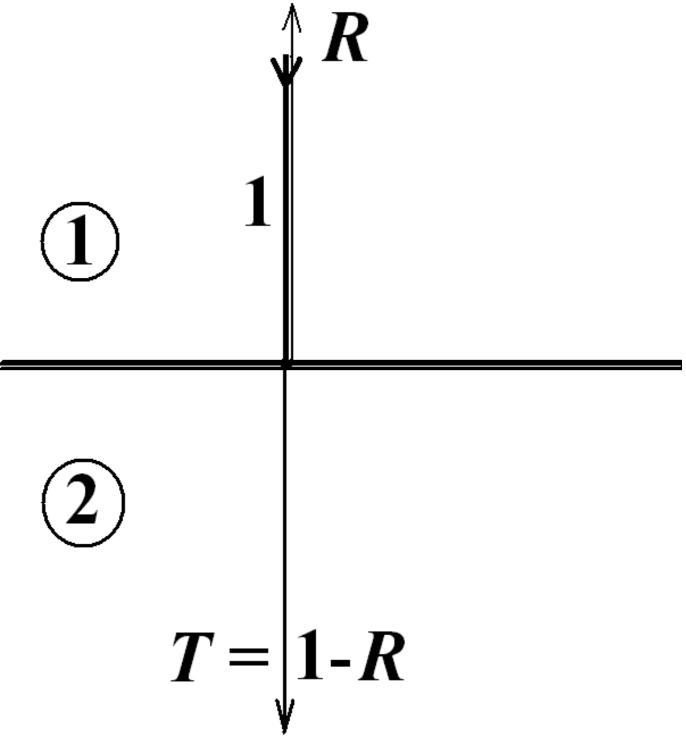


<table>
<tr><th></th><th>air</th><th>water</th><th>skin</th></tr>
<tr><td>c (m/s)</td><td>345</td><td>1480</td><td>1950</td></tr>
<tr><td>acoustic density</td><td>large</td><td>small</td><td>very small</td></tr>
<tr><td rowspan="2">critical angle, α_{crit} (degree)</td><td colspan="2">13.5</td><td></td></tr>
<tr><td></td><td colspan="2">49.4</td></tr>
</table>

Special case: perpendicular incidence

$$R = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1} \right)^2$$

$Z = \rho \cdot c$ is the *acoustic impedance (resistance)* of the medium, where ρ is the mechanical density and c is the speed of the wave. Reflection can occur at the boundary of two media of different acoustic impedances only: $R \neq 0$, if $Z_1 \neq Z_2$.



Example. Ultrasound is directed from air ($Z = 0.43 \cdot 10^3$ kg·m^{-2}·s^{-1}) perpendicularly to soft tissues ($Z = 1.6 \cdot 10^6$ kg·m^{-2}·s^{-1}). The reflected portion is $R = 0.9994$, *i.e*. the transmission is $T = 0.06\%$ only. If, however, water-base cellulose jelly as *acoustic coupling agent* ($Z = 1.5 \cdot 10^6$ kg·m^{-2}·s^{-1}) is used between the transducer and the soft tissue, then the reflected fraction diminishes to $R = 0.001$, *i.e*. the vast majority of the wave ($T = 0.999$) invades the tissue. The loss is at least 3 orders of magnitude if no appropriate acoustic coupling is applied.

Energy of the harmonic mechanical waves

The time average of the *energy density* of a harmonic wave:

$$\overline{w} = \frac{E_{\text{idoátlag}}}{\Delta V} = \frac{1/2 \cdot \Delta m A^2 \omega^2}{\Delta V} = \frac{1}{2} \rho A^2 \omega^2$$

Radiation power :

$$P = \overline{w} q c$$

Intensity (*power density*):

$$I = \frac{P}{q} = \overline{w}\, c = \frac{1}{2} \rho\, c\, A^2\, \omega^2 = \frac{1}{2} \rho\, c\, \mathrm{v}_{\max}^2 = \frac{1}{2} \frac{p_{\max}^2}{\rho\, c}$$

Example. Ultrasound of intensity 100 mW/cm^2, amplitude 2 nm and frequency 3 MHz is traveling in water (density 10^3 kg/m^3 and speed 1480 m/s). The maxima of the speed, acceleration and pressure are v_{max} = 3.7 cm/s (small), a_{max} = 7·10^4 g (extremely large!) and p_{max} = 0.5 bar (moderately large), respectively.

Distance-dependence of the intensity

The fronts of the waves emitted by point-like source in homogeneous and isotropic medium are concentric spheres. The energy emitted by the source within unit time, P is equal to the total energy transmitted through the spherical surface $q = 4\pi r^2$. The intensity

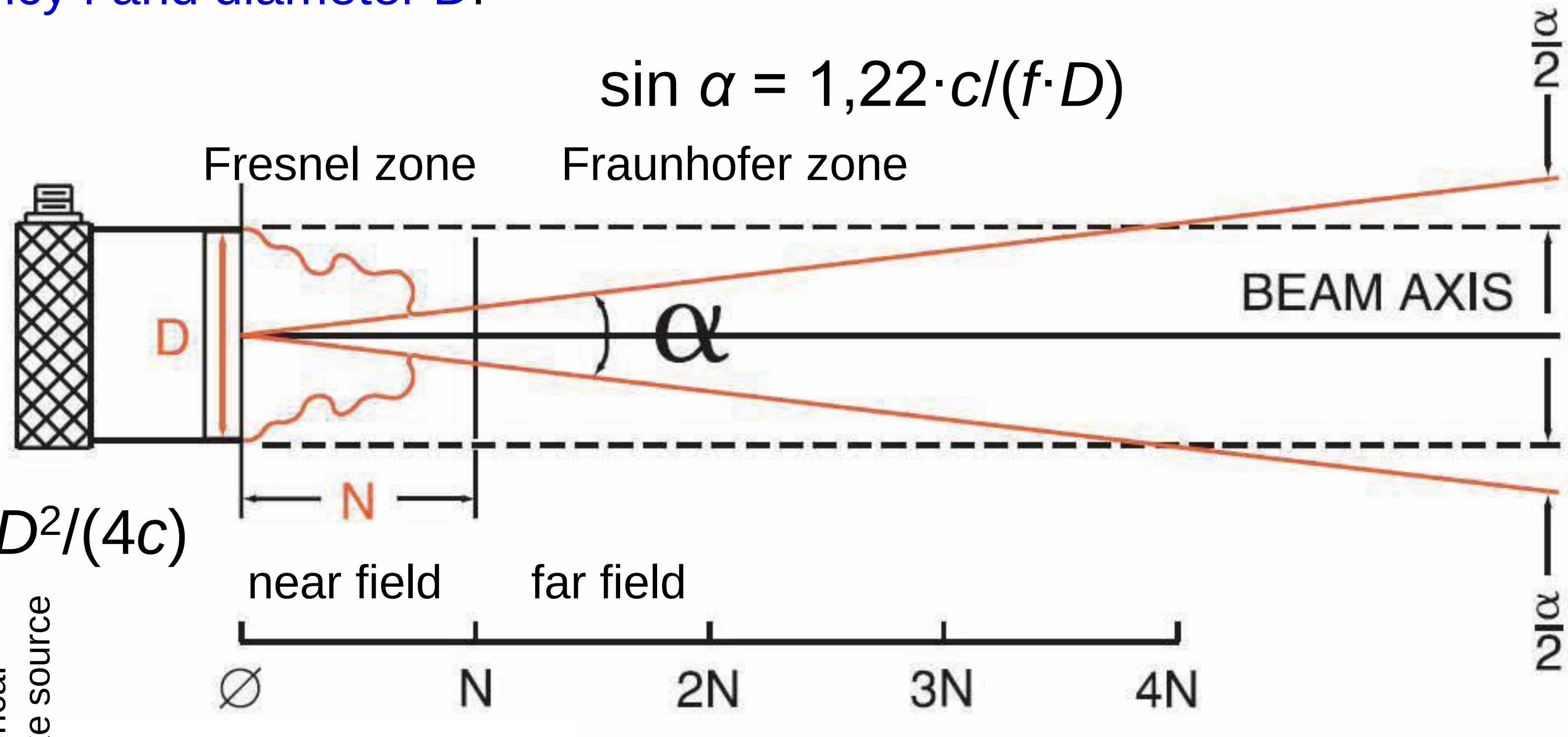
$$I = \frac{P}{q} = \frac{P}{4\pi r^2}$$

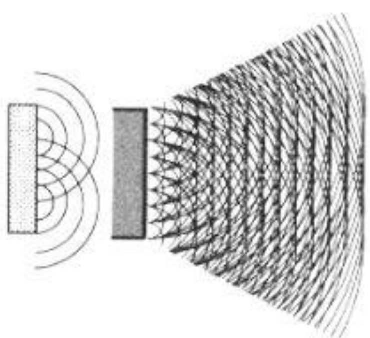
Two point-like sources of sound, patterns of waves and interference.

The distance- and direction-dependence of the intensity of an ultrasound of frequency f and diameter D:

$$\sin\alpha = 1{,}22 \cdot c/(f \cdot D)$$

$$N = f \cdot D^2/(4c)$$



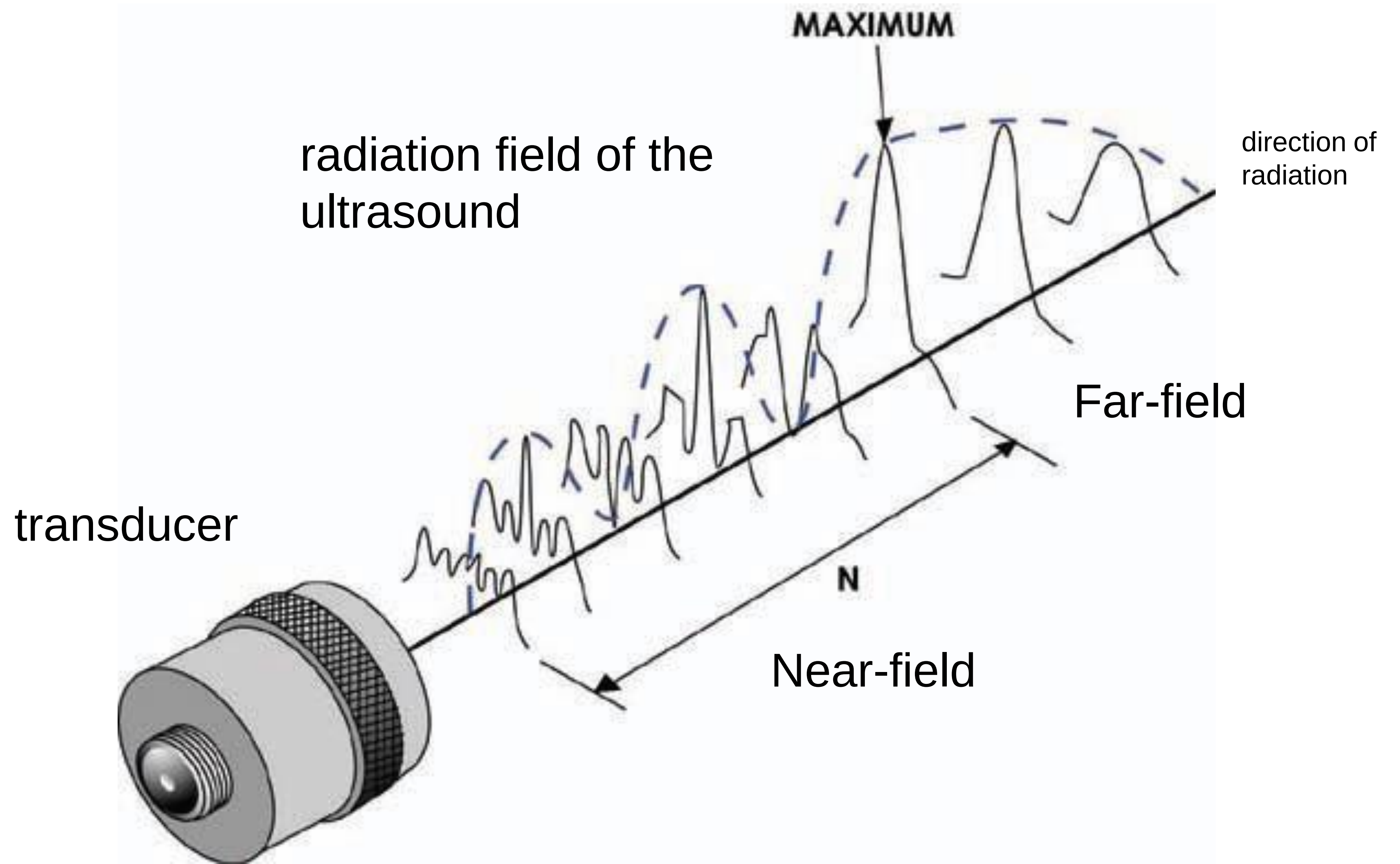


Interference of spherical waves from point-like source

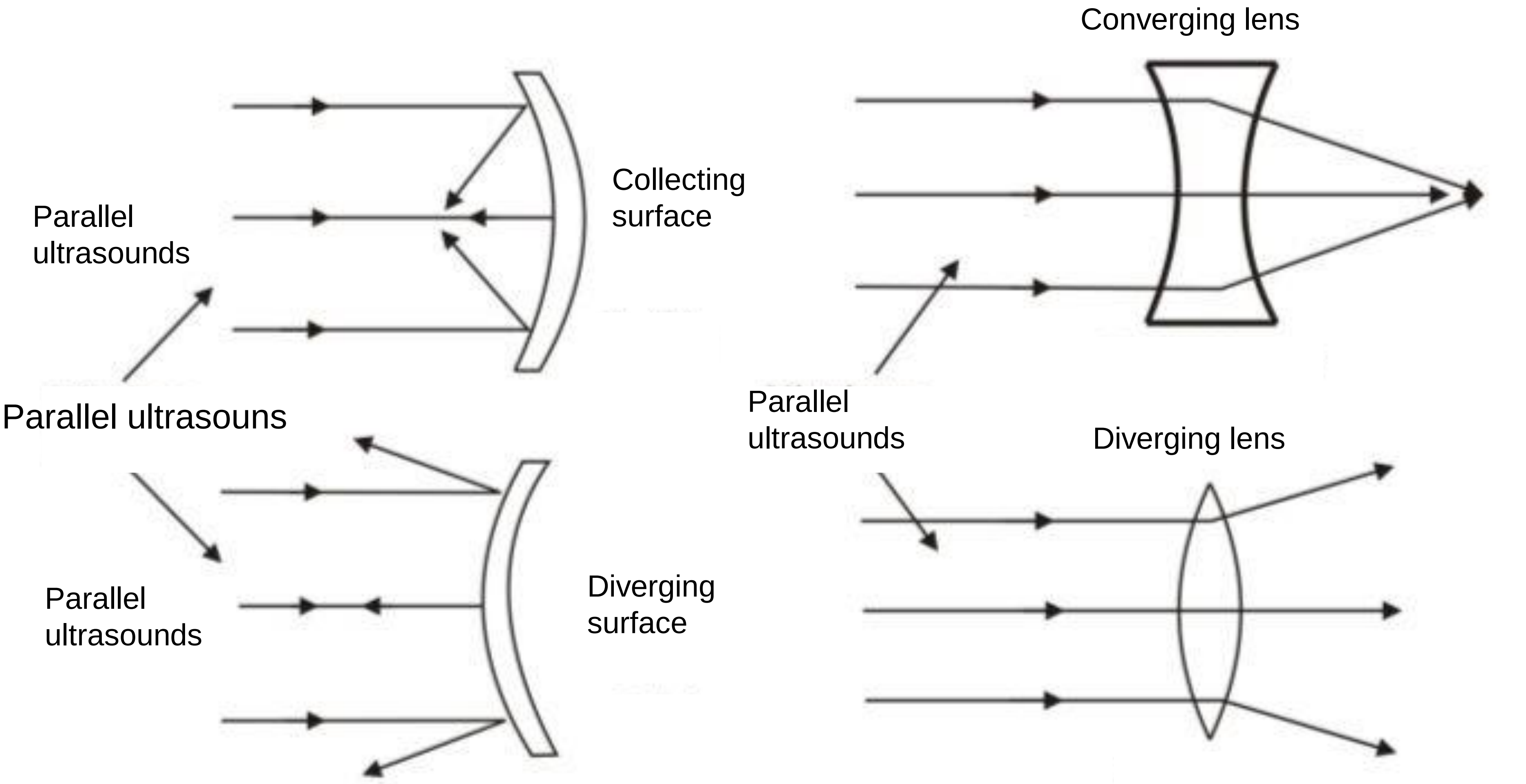
Example. The speed of ultrasound in soft tissue is c = 1580 m/s. The diameter of the transducer is D = 1 cm! The calculated and coupled values are included in the table.

f (MHz)	N (cm)	α (degree)
1	1.6	12.3
2	3.2	6.1
5	7.9	2.5

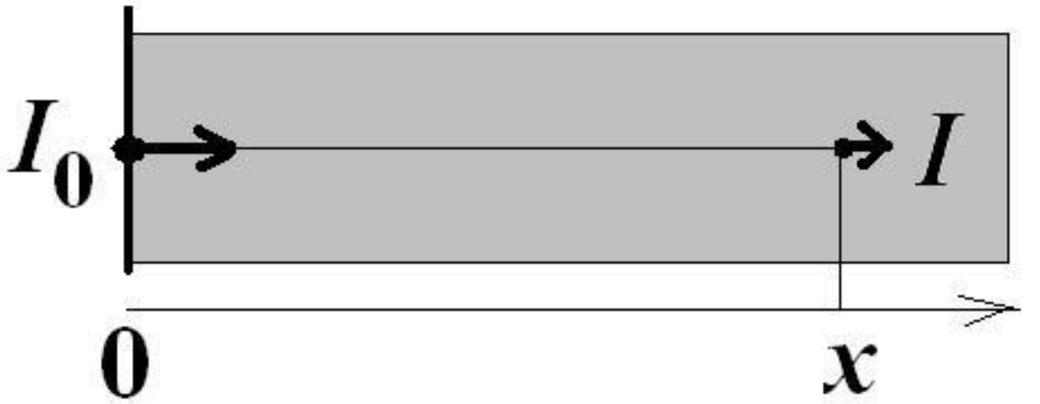
Spatial variation of the intensity perpendicular to the axis of propagation



Focussing the ultrasound:
acoustic method: mirrors and lenses
electronic way: phase control



Attenuation of ultrasound: the Beer's law of extinction



The intensity of the ultrasound $I(x)$ penetrating the medium with intensity I_0 will decrease exponentially according to the depth x due to losses as absorption, scattering etc.:

$$I = I_0 \exp(-\alpha \cdot x)$$

Here α is the sum of attenuation coefficients of all types of losses.

Objective sound intensity

Source of sound	P (W)
normal talk	10^{-5}
shout	10^{-3}
piano (maximum)	0.1
horn (car)	5
laud-speaker	10^{2}
horn for anti-aircraft defense	10^{3}

$$n = 10 \cdot \lg \frac{P_2}{P_1}$$ *decibel* (dB)

Subjective sound intensity, audibility, loudness

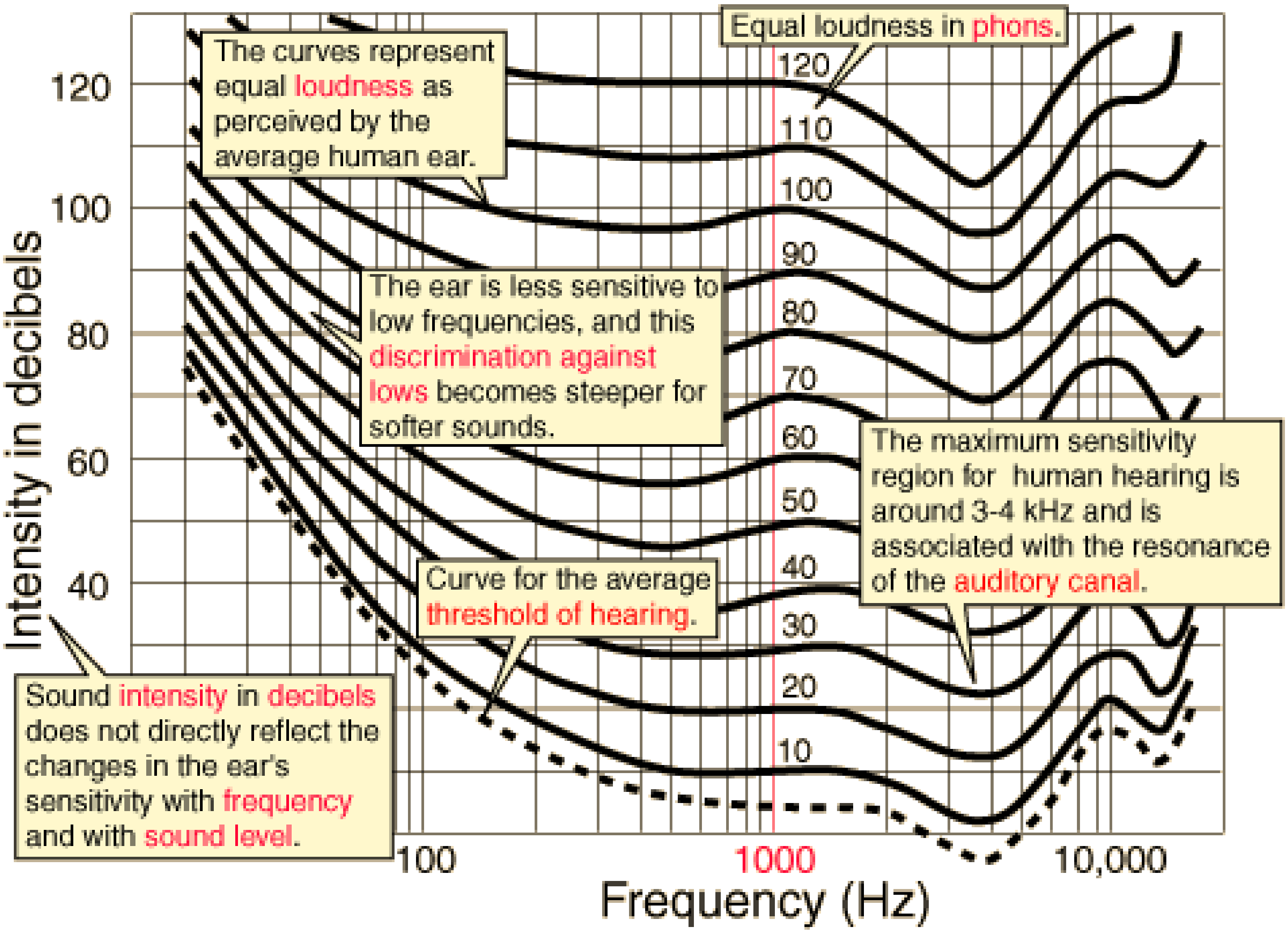
Source of sound	Subjective sound intensity (*phon*)
lower limit of audibility	0
rustle of leaf	10
whisper	20
noise of silent street	30
normal talk	50
shout	80
close to howling of the lion	120
upper limit of audibility, threshold of pain	130

$$H = 10 \lg \frac{I}{I_0} \quad \text{phon}$$

$$H\ (\text{phon}) = H_{1\ \text{kHz}}\ (\text{dB})$$

Subjective sound intensity, audibility, loudness

$$H\,(fón) = H_{1\ kHz}\,(dB)$$



How the speed of the sound depends on the properties of the medium?

In homogeneous and isotropic solid medium of infinite size, both longitudinal and transverse waves can propagate with speeds of

$$c_{\text{long}} = \sqrt{\frac{E}{\rho} \cdot \frac{1-\mu}{(1+\mu)(1-2\mu)}} \qquad c_{\text{trans}} = \sqrt{\frac{E}{\rho} \cdot \frac{1}{2(1+\mu)}}$$

Poisson's number :

$$\frac{c_{\text{long}}}{c_{\text{trans}}} = \sqrt{\frac{2(1-\mu)}{1-2\mu}}$$

$$\mu = -\frac{\Delta d / d}{\Delta l / l}$$

Here *E* is the Young's modulus (see the Hooke's law of the elasticity), *ρ* is the density and *μ* is the so called *Poisson's number* that expresses the transverse (cross) compression (Δ*d*/*d*) upon (longitudinal) elongation (Δ*l*/*l*). The Poisson's number is between 0 and ½ , typically between 0.3 and 0.4.

For many substances $\mu \approx 1/3$, therefore $c_{\text{long}} \approx 2 \cdot c_{\text{trans}}$. Generally, as $\mu \leq ½$, the longitudinal waves propagate faster than the transverse waves in the same solid medium. That can be utilized to localize the epicenter of an earthquake.

How the speed of the sound depends on the properties of the medium?

In (infinitely) long and (infinitely) thin elastic solid rode, transverse wave is not produced, thus longitudinal wave propagates. As there is no cross-compression ($\mu = 0$), the expression of the speed of the longitudinal wave reduces to:

$$c_{\text{long}} = \sqrt{\frac{E}{\rho}}$$

E is the Young modulus

In fluids

$$c = \sqrt{\frac{K}{\rho}}$$

K is the compression modulus of the fluid:

$$K = -\frac{p}{\Delta V / V}$$

In gases

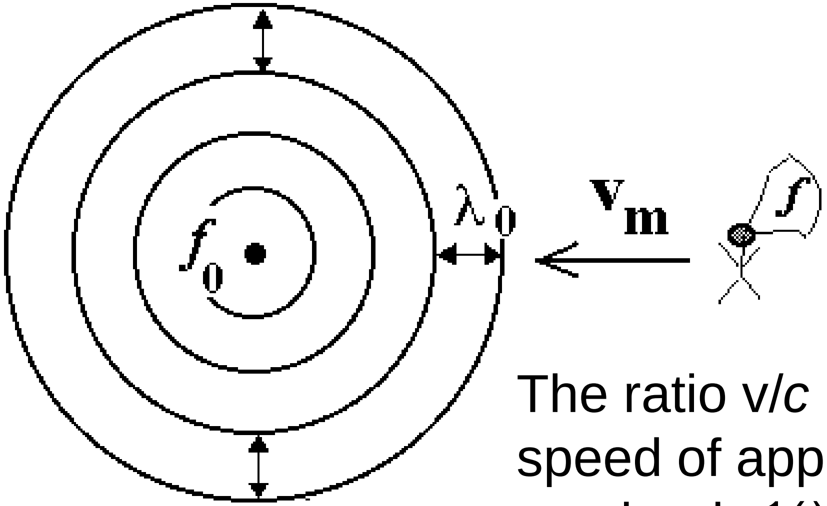
$$c = \sqrt{\frac{\kappa \cdot p}{\rho}} \quad \text{ideális gáznál} \quad c = \sqrt{\kappa \cdot RT}$$

where $\kappa = c_p/c_V$ is the ratio of the *specific heat capacities* of the gas under constant pressure and volume, respectively, *R* is the universal gas constant and *T* denotes the absolute temperature (*Laplace's expression*, 1816). In comparison, *E* (in solid states) can be formally replaced by *K* (in fluids) or by $\kappa \cdot p$ (in gases) in the expressions of the speed of longitudinal waves.

The Doppler-effect

a) The source is resting and the receiver is moving.

If the receiver (man) is moving towards the resting source with velocity v_m, then it will detect not only f_0 vibrations within 1 s but additionally $v_m/\lambda_0 = v_m \cdot f_0/c$ more vibrations. Therefore, the approaching (+) or receding (-) receiver will observe the emitted f_0 frequency as



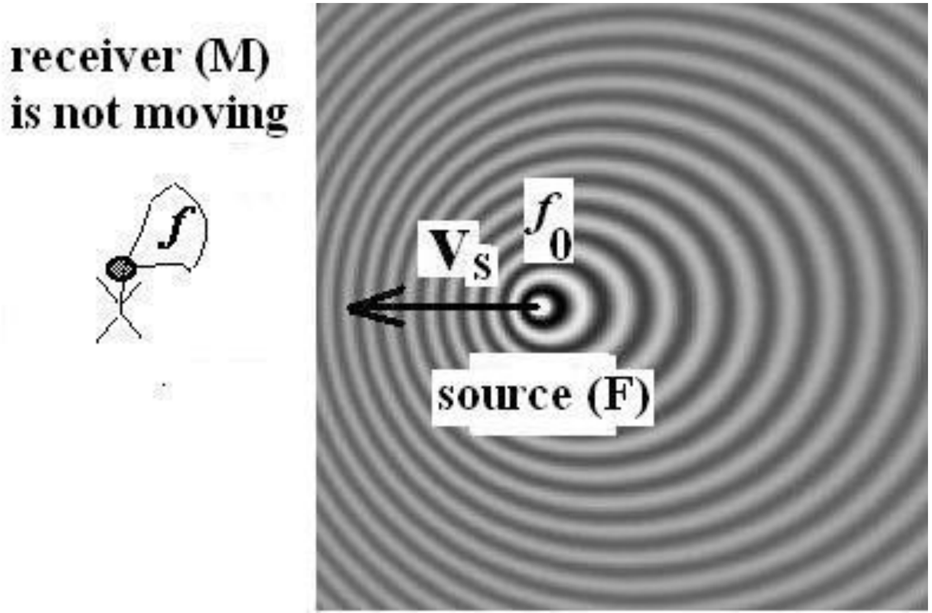
$$f = f_0\left(1 \pm \frac{v_m}{c}\right)$$

The ratio v/*c* is called the *Mach number*. For example, at speed of approach (removal) of $v_m = ½\, c$ (the Mach's number is ½), the frequency of the observed sound will double (half), *i.e*. the pitch level increases (decreases) by one octave.

The Doppler-effect

b) The source is moving and the receiver is at rest.

The source (emitter, F) moving with speed v_s to the receiver will emit the first phase of the oscillation at $t = 0$ and the last phase at $t = T_0$ when the source gets closer to the receiver by a distance $v_s \cdot T_0$. Therefore, the wavelength becomes shorter by $v_s \cdot T_0$ in front of the receiver. The new wavelength is $\lambda_0 - v_s \cdot T_0$. Because the shorter waves propagate in the resting medium with unchanged speed (c), the observed frequency is $f = c/(\lambda_0 - v_s \cdot T_0)$. If the source of sound approaches to (-) or moves away from (+) the resting receiver, the observed frequency can be given in the following expression:

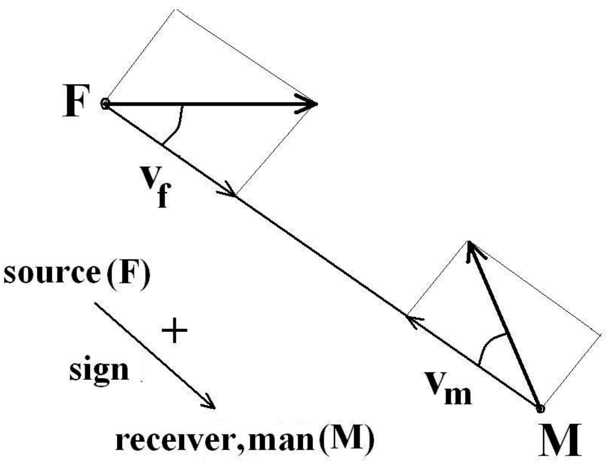


$$f = \frac{f_0}{1 \mp \frac{v_s}{c}}$$

The Doppler-effect

The unified expression for both movements:

$$f = f_0 \frac{1 - \frac{v_m}{c}}{1 - \frac{v_f}{c}}$$



where v_f and v_m are the projected speed vectors of the source *F* and the receiver *M* to the interconnecting straight line *FM*. Both velocities should be measured relative to the medium. If the projected component of the vector shows to the $F \rightarrow M$ direction, then the sign is positive, if it is opposite, then the sign of the speed component is negative. Of course, if the FM distance remains constant (e.g. the movement is perpendicular to this direction), then the motion does not result in Doppler-shift of the frequency.

The optical Doppler shift (red shift)

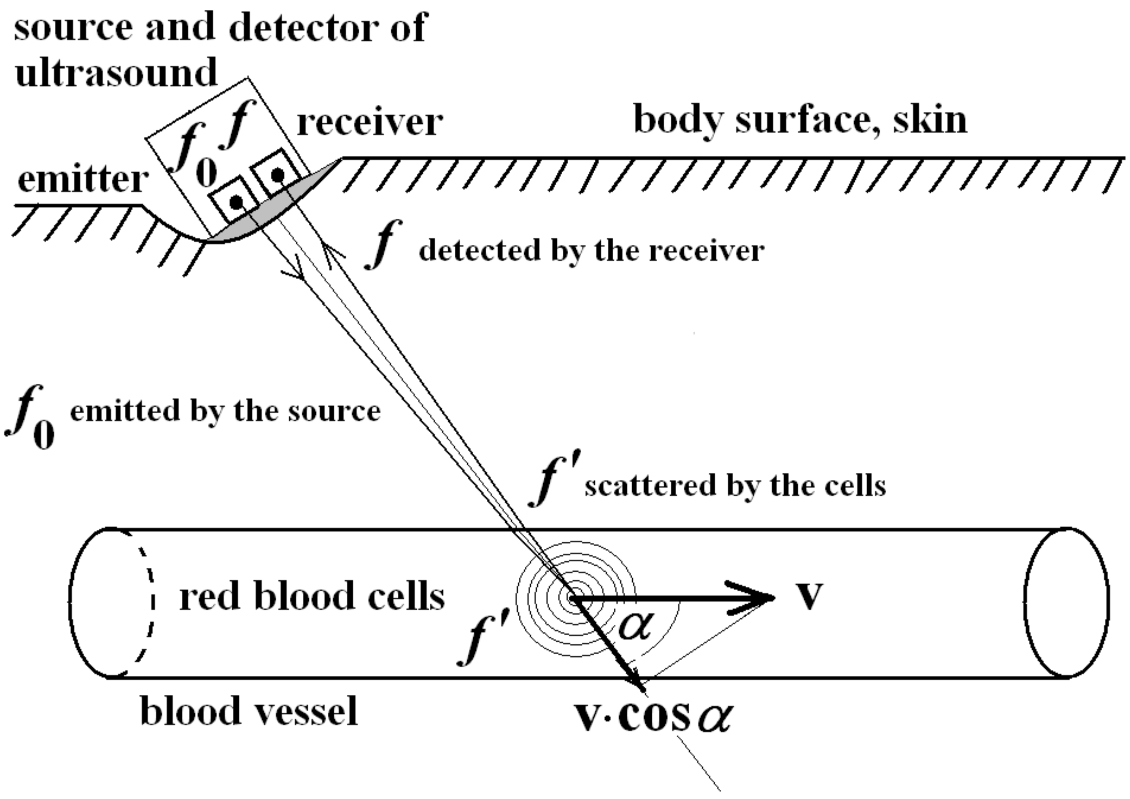
The physical background of the optical Doppler-shift differs from that of the acoustic version because no medium (no “ether”) is needed to mediate the optical waves (light). In the optical Doppler effect the medium plays no role and therefore the Doppler shift is determined by the relative speeds of the emitter and the receiver. The observed frequency can be calculated from the relativity theory (more precisely from the Lorentzian transformation):

$$f = f_0 \frac{1 \pm \frac{v}{c}}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} \qquad \text{if} \quad v << c, \quad \text{then} \quad f = f_0 \left(1 \pm \frac{v}{c}\right)$$

Blood velocity measurement based on Doppler shift of ultrasound

The Doppler shift $f = f' \frac{1}{1 + \frac{\mathrm{v} \cdot \cos a}{c}}$ $f' = f_0 \left(1 - \frac{\mathrm{v} \cdot \cos\alpha}{c}\right)$

$$\Delta f = f - f_0 = -2 f_0 \frac{\frac{\mathrm{v}\cos\alpha}{c}}{1 + \frac{\mathrm{v}\cos\alpha}{c}}$$



As the velocity of the red blood cells (vvt) is orders of magnitude smaller than the speed of propagation of the sound (v << c), then

$$\Delta f = -2 f_0 \frac{\cos\alpha}{c} \cdot \mathrm{v}$$

Optimum of the frequency of the ultrasound to measure the velocity of the blood

The Doppler-shift is proportional to the frequency of the ultrasound. The higher is the frequency, the larger will be the Doppler-shift and the more precise will be the determination of the velocity of the red blood cell:

$$\Delta f = \text{const}_1 \cdot f$$

The intensity of the echo from a red blood cell from depth *d* is

$$I = \text{const}_3 I_0 \exp(-\alpha \cdot 2d)$$

the intensity of the reflected wave (echo) will show opposite tendency. In soft tissues and in frequency range (2-20 MHz) used in medical diagnostics, the loss factor α (the sum of the absorption, scattering, etc.) in the exponent of the exponential extinction law increases proportionally with increase of the frequency:

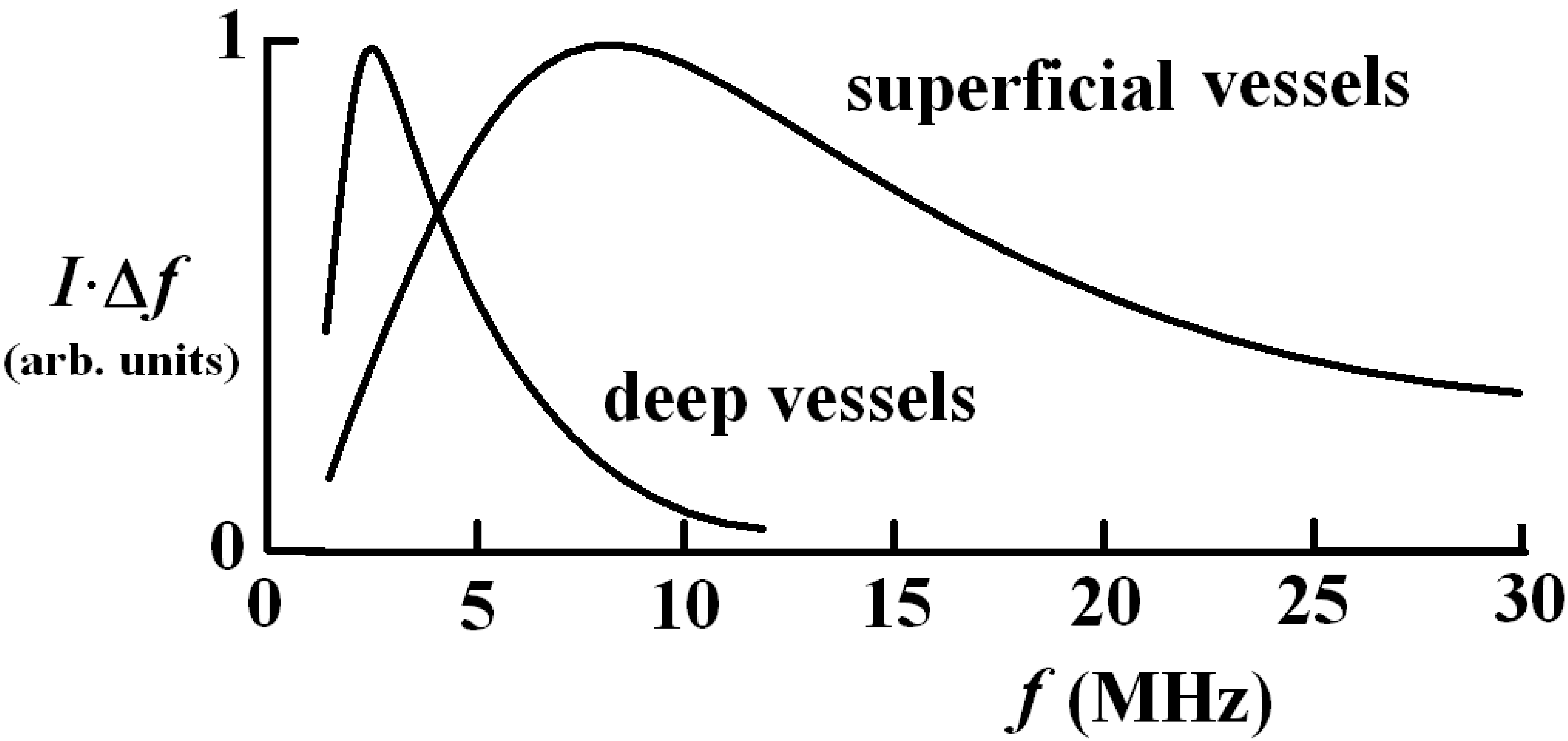
$$\alpha = \text{const}_2 \cdot f$$

We will consider the frequency as optimum (f_{opt}) if the $I \cdot \Delta f$ product is the largest (maximum). The necessary condition is the disappearance of the first derivative of $I \cdot \Delta f$ according to the frequency f : $d(I \cdot \Delta f)/df = 0$, which gives

$$f_{opt} = \frac{1}{2d \cdot \text{const}_2}$$

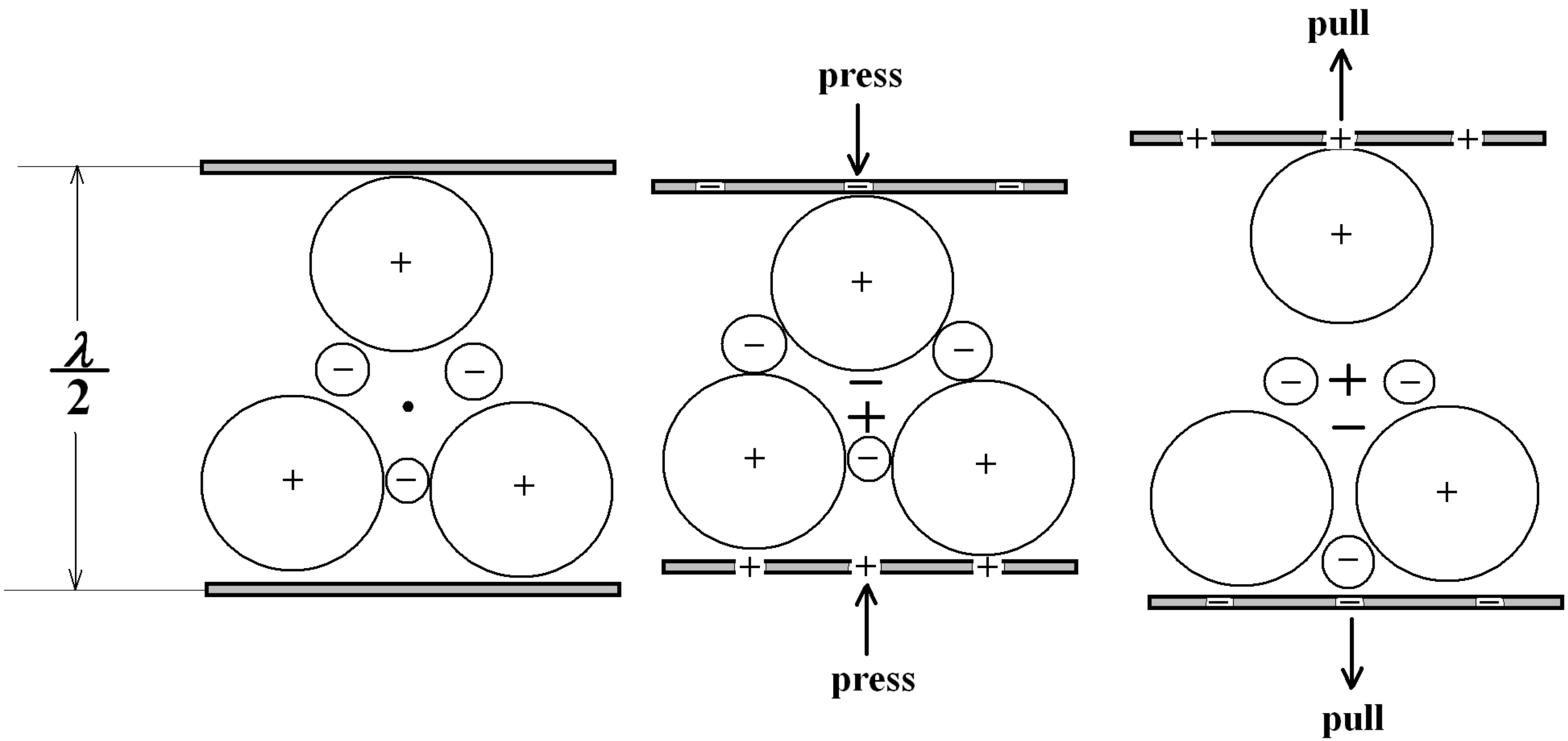
Optimum of the frequency of the ultrasound to measure the velocity of the blood

$$f_{\text{opt}} = \frac{90\ \text{MHz} \cdot \text{mm}}{d}$$



Production of ultrasound

by inverse piezoelectric phenomenon



If the polarity of the electric potential at the plates is alternating with frequency *f*, then the crystal will perform forced oscillation with the same frequency. If this frequency coincides with the self frequency of the crystal, then the amplitude of the vibration will be enhanced by resonance. A layer thickness of $\lambda/2$ (= $c/2f$) will assure the condition of resonance: the fundamental frequency will be generated by standing wave with nodes at the electrodes.

Medical application of ultrasound

f = 1 MHz Maximum values in water	I =10 mW/cm^2 diagnostics	I = 3 W/cm^2 therapy	Shock waves
displacement: $x = \frac{\sqrt{2I/Z}}{\omega}$	2 nm	35 nm	The high intensity shock waves (pulses) consist of one half waves only, therefore the expressions derived for harmonic (continuous) waves cannot be applied without restrictions.
relative elongation: $x \cdot \frac{2\pi}{\lambda} = \frac{\sqrt{2I \cdot Z}}{E}$	$8{,}4 \cdot 10^{-6}$	$1{,}47 \cdot 10^{-6}$	
acceleration: $x \cdot \omega^2 = \omega \sqrt{\frac{2I}{Z}}$	$2 \cdot 10^3$ m/s^2	$3{,}5 \cdot 10^4$ m/s^2	
pressure of sound: $E\ x \frac{2\pi}{\lambda} = \sqrt{2I \cdot Z}$	$2 \cdot 10^4$ Pa	$3{,}5 \cdot 10^5$ Pa	40 MPa (!)

Problems for home works and/or seminars

1) How much is the wavelength of the normal tone "*a*" („from Vienna", 440 Hz) in the air and in the water? Give the similar values for the "Hungarian" tone "*a*" (435 Hz)! What would be the experience of the audience if the orchestra tuned their instruments to these different notes?

2) The octave of a "wohl-temperiertes Klavier" consists of 12 notes whose frequencies follow a geometrical series (the ratios of the neighboring frequencies are equal). The evenly tuned scale was introduced in the music by the Bach's works from 1720. How much is the frequency of the tone "*c*" if the scale is tuned to the normal tone "*a*" (440 Hz)?

3) What are the fundamental frequencies of the open and closed pipes of length 80 cm?

4) What frequencies will be specifically amplified in the external auditory canal of the human ear of length 2.5 cm? Does this effect increase or decrease the threshold of hearing?

5) The whales are very sensitive to underwater waves of low frequency. How long could be their external auditory canal if the maximum of hearing sensitivity is 100 Hz?

6) A porpoise sends an echolocating pulse (60 kHz) as it tracks the path of a shark. The power of the pulse is 30 mW. The intensity of the pulse at the position of the shark is $1.5 \cdot 10^{-5}$ W/m^2. (a) What is the distance between the shark and the porpoise? (b) What is the displacement amplitude of the water molecules adjacent to the shark?

7) Dogs have very sensitive hearing. Suppose their threshold of hearing is $1 \cdot 10^{-15}$ W/m^2. If a sound is judged by a human to be 50 dB, what is the correct dB rating for a dog?

Problems for home works and/or seminars

8) Three loudspeakers each of 20 dB intensities are operating at the same time. How much is the resultant sound intensity?

9) The middle ear amplifies the pressure of sound by 20 times (the intensity of the sound by 400 times) during transmission from the ear drum of the external ear to the oval window of the inner ear. What would be the increase of the hearing threshold (in dB) if the middle ear failed this function?

10) A train is running through the railway station with 10 m/s speed. The frequency and intensity of the steam whistle of the locomotive are 1 kHz and 100 dB, respectively. Standing at the platform 1 m away from the rail, what would be the drop of intensity and observed frequency (pitch of the tone) 5 s after the locomotive passed by?

11) A bat emitting cries at 80 kHz flies directly at a wall. The frequency it hears is 83 kHz. How fast is it flying?

12) What is the radial speed of a star whose spectrum shows the wavelength of the Na spectral line of 589.6 nm at 592.0 nm? The speed of the light in vacuum is 3·108 m/s.

13) Two sound waves of equal amplitudes propagate in the same direction. Their wavelengths in the air are 72.0 cm and 77.2 cm. Do we observe beats?

14) A circular ultrasound transducer of 1 cm diameter is operating in water at 1 MHz frequency. How much will be the diameter of the ultrasound beam at 4 cm from the transducer?

15) What is the angle of refraction of the ultrasound that arrives at 12^o angle of incidence to the boundary of air (c_{air} = 343 m/s) and muscle tissue (c_{muscle} = 1590 m/s)?

16) What is the ratio of reflection and transmittance of the ultrasound at the boundary of muscle tissue ($Z = 1.7 \cdot 10^6$ kg·m^{-2}·s^{-1}) and fat ($Z = 1.35 \cdot 10^6$ kg·m^{-2}·s^{-1})?
17) Do the air bubbles in water collect or disperse the (parallel) ultrasound beams? Do they act as converging or diverging lenses?
18) Estimate the penetration depths of the ultrasound of frequency 1 MHz in lung (α = 7 cm^{-1}), in bones (α = 3 cm^{-1}), in muscles (α = 0.3 cm^{-1}) and in blood (α = 0.03 cm^{-1})! The sum of the losses due to absorption, scattering etc. (total extinction coefficient) is denoted by α, and the actual values are in the brackets.
19) A 10 cm depth section of the liver is investigated by ultrasound of frequency 1 MHz and intensity 1 W/cm^2. The radiation lasts for 10 s. Estimate the temperature increase of the area! The sum of the loss coefficients in the liver is α = 0.17 cm^{-1}, and replace the liver by water (from thermal points of view) of 4.2 J/gK specific heat capacity.
20) The proper position of the plastic lens after cataract removal can be checked by ultrasound-echo experiment ("A"-image). The transducer is attached to the cornea and the echo from different layers of the eye bulb is monitored on the screen of an oscilloscope. The following signals can be visualized: „A" – initial echo that originates from reflection from the contact fluid between the transducer and the cornea, „B" – double echo that comes from the two boundaries of the cornea (difficult to separate), „C" and „D" – echo from the two surfaces of the lens and „E" – echo from the back wall of the eye bulb. How much is the length of the bulb if the (time) gap between the „B" and „E" echo amounts to 30 μs? The speed of the sound is 1600 m/s.
21) The cataract is emulsified by low frequency (23 kHz) and large intensity (1 kW/cm^2) ultrasound (produced by magnetostriction) followed by drawing through a cut made between the cornea and the sclera. What should be the amplitude of the ultrasound? The acoustic impedance of the cataract is $1.75 \cdot 10^6$ kg·m^{-2}·s^{-1}.