Mechanical oscillation, resonance

Lecture for freshman (first year) medical students

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Preparation for exams

Lecture (slide show, see the home page of the Institute)

Handout (more text, see the home page of the Institute)

Seminars

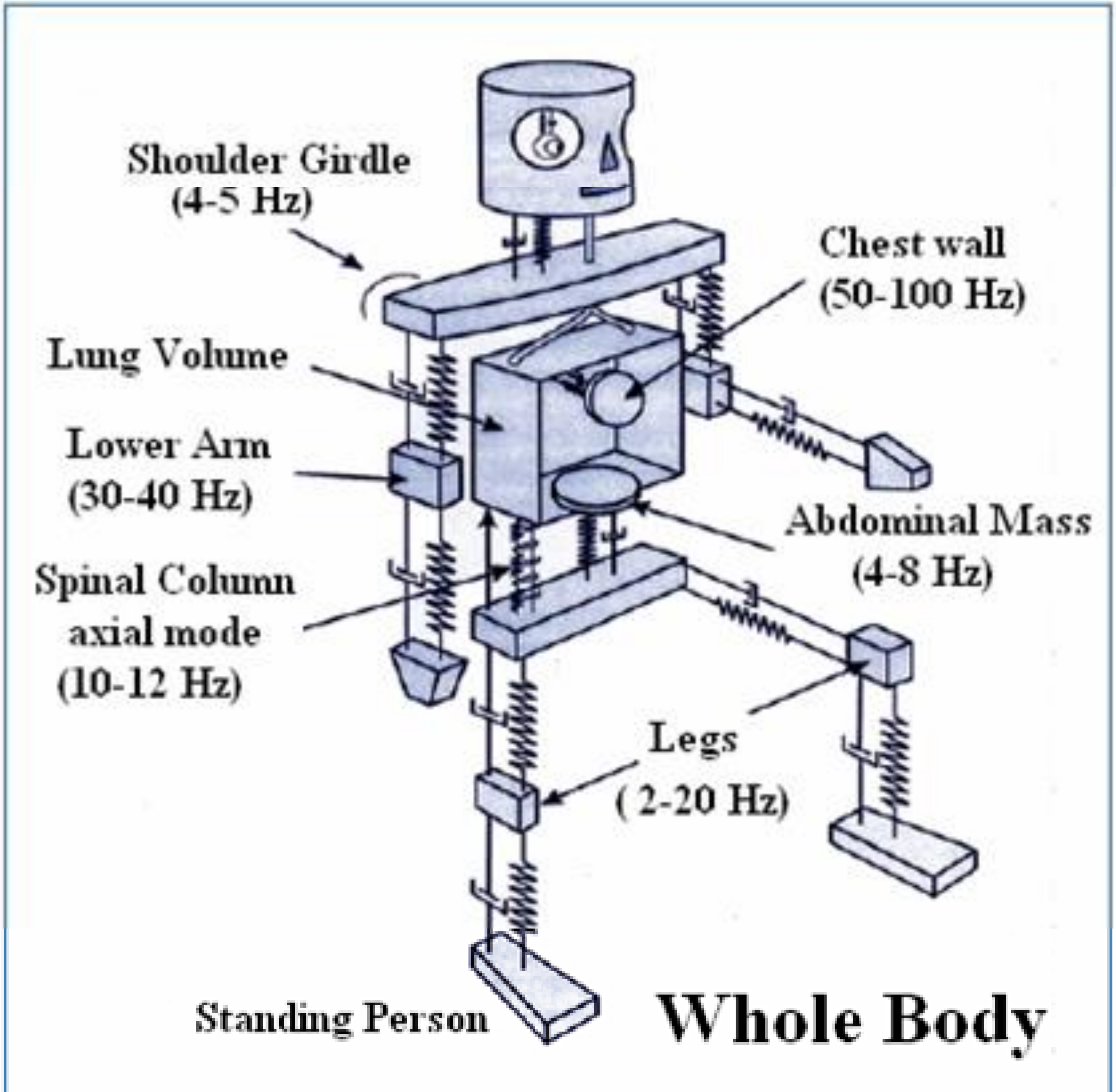
Importance of ability to solve problems: try to solve the medical physics oriented problems and be active in the seminars.

Suggested texts to consult

J. J. Braun: Study Guide: Physics for Scientists and Engineers, HarperCollinsCollegePublishers, New York 1995 or any other college physics texts.

P. Maróti, I. Berkes and F. Tölgyesi: Biophysics Problems. A textbook with answers, Akadémiai Kiadó, Budapest 1998.

S. Damjanovich, J. Fidy and J. Szöllősi (eds.): Medical Biophysics, Medicina, Budapest, 2009.



Mechanical oscillations in human bodies.

Extended network of springs (elastic elements) and dampers.

Forced and self oscillation

The frequencies of self oscillations are in the very low frequency range.

Medical consequencies of mechanical vibrations,

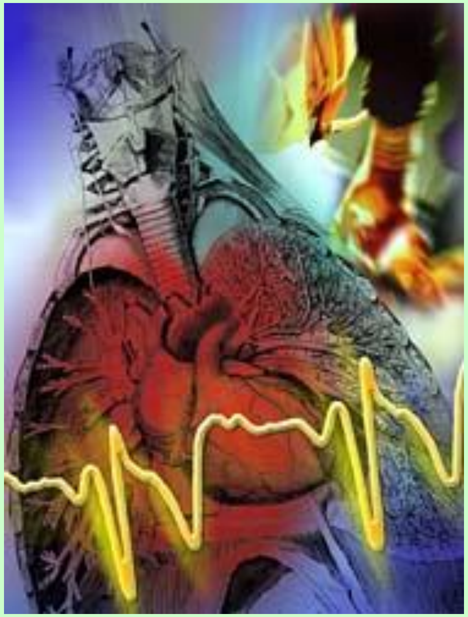
the periodic movement (oscillation) of the heart

Medical consequencies of mechanical vibrations,

percussio and auscultatio

As early as in 1761, *L. Auenbrugger* introduced the traditional method of *percussion* (*sounding*) in the medical investigation.

In Hungary, Ignác Sauer introduced this method into the every day’s routine of physicians.



Medical consequencies of mechanical vibrations,
tremor

Tremor: unintentional oscillation (vibration) of part of the body (mainly arms or hands).
More than 10 different types of tremors are identified: the most frequently occuring tremors are
- the physiological tremors,
- the increased physiological tremors,
- the essential tremors (ET), and
- the Parkinson-tremors (PT) manifested in Parkinson disease.

The frequency ranges of the tremors are characteristics but may be overlapped:
< 4 Hz cerebellar and Holmes-tremor,
4 – 6 Hz 80% of Parkinson tremor and 50% of ET,
6 - 11 Hz physiological tremor, 50% of essential tremors and 20% of tremor appearing in Parkinson-disease and
> 11 Hz orthostatic tremor.

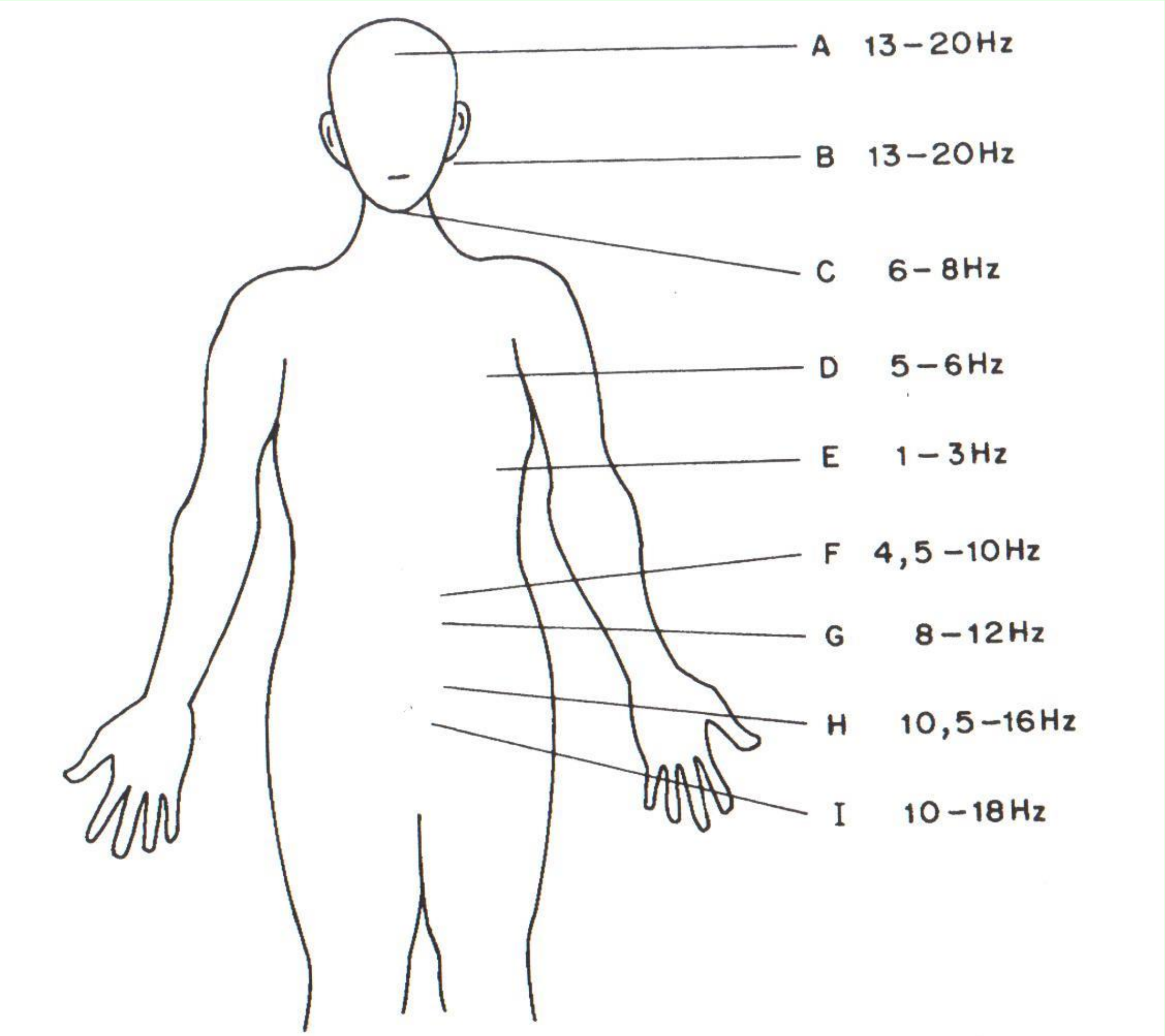
Medical consequencies of mechanical vibrations,
intervention radiology, lithotripsy

Endoscopic retrograde colangiopancreatography. Large stone in *ductus choledochus*.

The stone of *ductus choledochus* is in the basket of the intraendoscopic and mechanical litotriptor.

Breaking the stone in the intraendoscopic mechanical litotriptor: the basket became smaller.

Effects of mechanical (infra)oscillations on human body

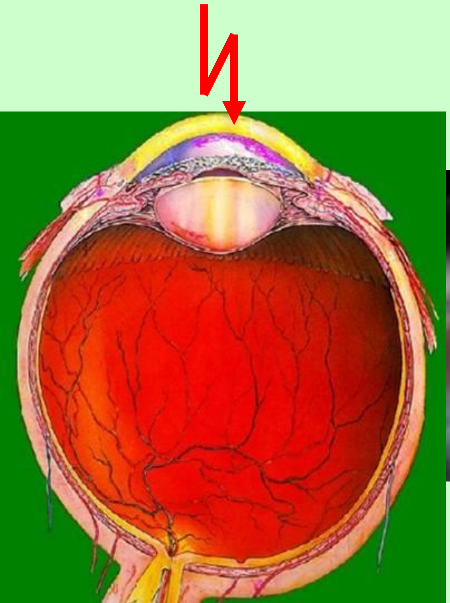


J.R. Cameron and J.G. Skofronick, 1978

Medical consequencies of mechanical vibrations,

laser operation in ophthalmology, KO

Mechanical vibrations (that lead to shock waves) are produced in the eye bulb if the breaking power of the cornea is designed to modify (“throw away your glasses”) by a series of high energy (UV) laser light pulses (graduate ablation (evaporation) of the layers of the cornea by intense laser flashes). It is called “sculpture of the cornea”. Similar physical phenomena arise when the head of a boxer is hit by a strong and sudden punch (*e.g*. KO). The liquid brain closed in the solid skull (cranium) is exposed to heavy oscillations and may suffer severe damages.



Medical consequencies of mechanical vibrations,

sport injuries, accidents

Athletes jumping up and down in competitive way (*e.g*. basketball or handball players) may suffer severe consequences (damage) due of vibrations evoked by large forces in their bodies.

Chronic overload of Achilles tendon.

Injuries of the bands.

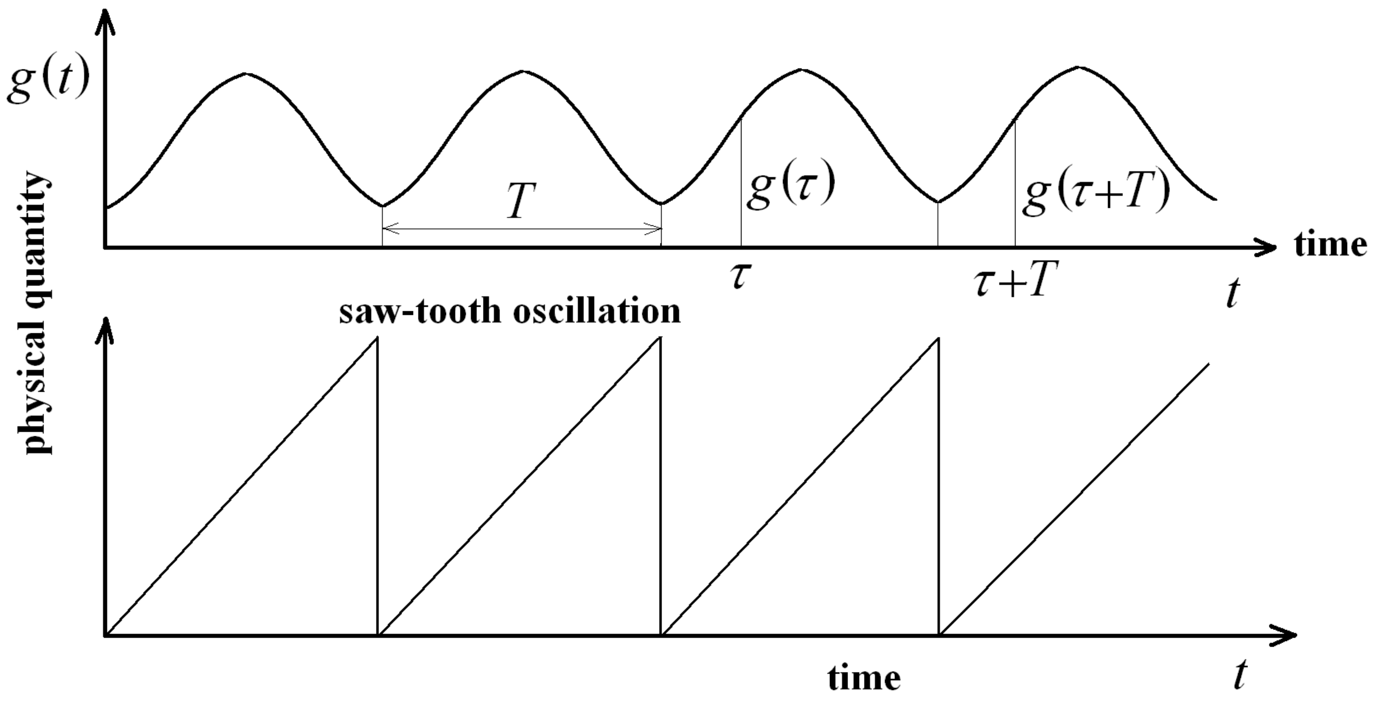
Fracture of bones.

Principal definition: a physical quantity makes *oscillation* (in strict (mathematical) sense of the word) if its value is periodic function of the time t:

$$g(t) = g(t+T)$$

Here $g(t)$ denotes the actual value of the physical quantity at time t. The *time of period T* is the shortest time interval after which the physical quantity takes the same value as it had at time t. Its reciprocal (inverted) value is called *frequency*: $f = 1/T$, its dimension is 1/time and its unit is 1/s = 1 Hz.

For example, if the heart beats 50 in a minute, then its frequency is $f = 50\ (\text{min})^{-1} = 50/60$ Hz and its time of period is $T = 60/50$ s.



More generally, we talk about oscillations (vibrations) even in the lack of strict periodic changes of the physical quantity *versus* time. If the periodic character of the physical quantity can be recognized, the motion is frequently called as oscillation (e.g. damped oscillation).

Classification: according to its actual mathematical form, the function $g(t)$ may include several types of oscillations.

Simple harmonic oscillation (*sine/cosine oscillation*):

$$g(t) = A \cdot \sin(\omega t + \alpha)$$

where A denotes the *amplitude*, ω is the *angular frequency* ($2\pi/T$), $\omega t + \alpha$ is the *phase* and α is the *initial phase*. (It is recommended to measure the phase here not in *degrees* but in *radians*.) The simple harmonic oscillation is described by a sole sine function of the time.

Anharmonic oscillation: the physical quantity performs simultaneously finite (see *e.g*. the Lissajous-curves) or infinite (see the Fourier-theorem) numbers of harmonic oscillations. Typical example is the *saw-tooth vibration.* If a slow and long inspiration is followed by a sudden expiration periodically then the chest volume describes a saw-tooth vibration.

Simple harmonic oscillation.

Kinematic description. We initiate from the position-time relationship and derive other physical quantities.

The position (*x* coordinate) vs. time:

$$x = A \cdot \sin(\omega t + \alpha)$$

The velocity:

$$\mathrm{v} = \frac{dx}{dt} = A\omega \cdot \cos(\omega t + \alpha)$$

has maximum while crossing the origin at times t = 0, T/2, T, ... and disappears (v = 0) at the turning points (amplitudes) of the oscillation at times t = T/4, 3T/4, ...

The acceleration:

$$a = \frac{d\mathrm{v}}{dt} = -A\omega^2 \cdot \sin(\omega t + \alpha) = -\omega^2 \cdot x$$

which is proportional to the deviation from the origin and shows in opposite direction therefore it is directed always to the equilibrium position (origin).

Simple harmonic oscillation.

Dynamic description. Substitute the position-time function of the movement in the principal law of the dynamics (Newton's 2nd law):

$$F = m \cdot a = -m\omega^2 \cdot x = -k \cdot x$$

The *directional force* (spring constant) :

$$k = m\omega^2 = m\left(\frac{2\pi}{T}\right)^2$$

The *time of period* of the harmonic oscillation:

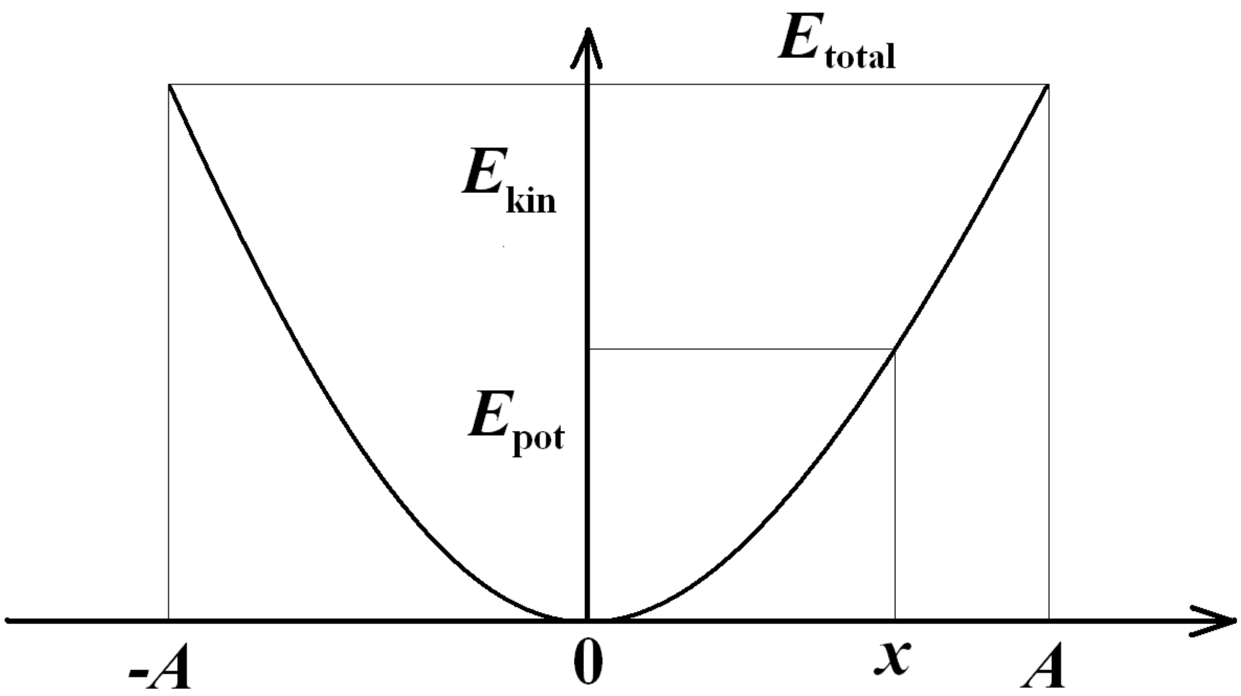
$$T = 2\pi\sqrt{\frac{m}{k}}$$

The time of period is directly proportional to the square root of the mass (if *k* is constant) and indirectly proportional to the square root of the directional force (if *m* is unchanged).

Energy of the mass under harmonic oscillation.

Because the field of force is conservative, the total mechanical energy (the sum of the kinetic and potential energies) will remain constant. The energy is not dissipated:

$$E_{\text{total}} = \frac{1}{2}m\text{v}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 A^2 = \text{constant}$$



Superposition of harmonic oscillations.

1. Addition (superposition) of two unidirectional vibrations.
   a) The frequencies are equal.

$$x_1 = A_1 \sin \omega t$$

$$x_2 = A_2 \sin(\omega t + \alpha_0)$$

The resultant oscillation is the simple (algebraic) sum of the components:

$$x = x_1 + x_2 = A \sin(\omega t + \alpha)$$

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos \alpha_0} \qquad \mathrm{tg}\,\alpha = \frac{A_2 \sin \alpha_0}{A_1 + A_2 \cos \alpha_0}$$

Special cases:
-If the phases are equal (the two component vibrations are “in phase”, $\alpha_0 = 0$), then the amplitudes of the components are summed up: $A = A_1 + A_2$, and the resultant phase constant coincides with that of the components: $\alpha = 0$. This is called *constructive superposition*.

- If the vibrations have opposite phases ($\alpha_0 = \pi$), then the difference of the amplitudes should be taken: $A = |A_1 - A_2|$, and the resultant phase is equal to the phase of the component of larger amplitude. If $A_1 = A_2$, then $A = 0$, *i.e*. the two vibrations cancel (quench) each other (see later the similar effect of *interference* of waves of sound or light). This phemonenon is called *destructive superposition*.

Superposition of harmonic oscillations.

b) The frequencies are different.

$$x_1 = A_1 \sin \omega_1 t$$

$$x_2 = A_2 \sin(\omega_2 t + \alpha_0)$$

The resultant oscillation cannot be taken to the form of

$$x = x_1 + x_2 = A \sin(\omega t + \alpha)$$

consequently it is not a harmonic oscillation. In addition, it is even not a periodic motion. The resultant motion will be periodic only if the ratio of the frequencies of the two components (ω_1/ω_2) is a rational number. If this is the case, then $\omega_1 = n_1 \cdot \omega$ and $\omega_2 = n_2 \cdot \omega$ (n_1 and n_2 are relative prime integer numbers), and the values of the function

$$x = A_1 \sin(n_1 \omega t) + A_2 \sin(n_2 \omega t + \alpha_0)$$

will be repeated in a time of period $T = 2\pi/\omega$.

Superposition of harmonic oscillations, the *beats*.

For the sake of simplicity,
- the amplitudes of the two oscillations are the same and
- the phase difference is zero:

$$x_1 = A\sin\omega_1 t$$

$$x_2 = A\sin\omega_2 t$$

The resultant oscillation is

$$x = x_1 + x_2 = 2A\cos\frac{\omega_1 - \omega_2}{2}t \cdot \sin\frac{\omega_1 + \omega_2}{2}t$$

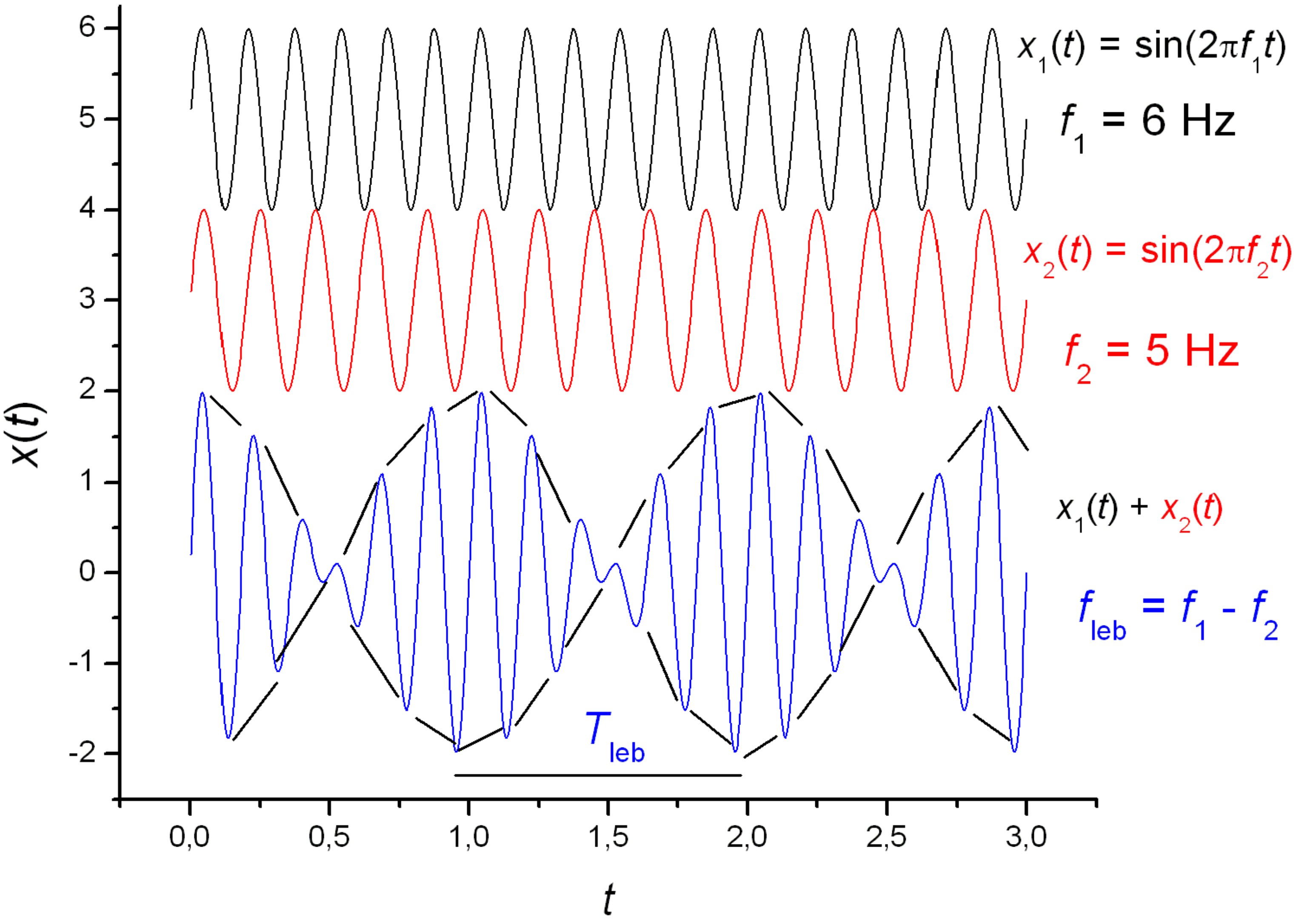
The *period of beat*

$$T_{\text{beat}} = \frac{2\pi}{\omega_1 - \omega_2} = \frac{1}{f_1 - f_2}$$

The *frequency of beat*

$$f_{\text{beat}} = f_1 - f_2$$

The beats.



$x_1(t) = \sin(2\pi f_1 t)$
$f_1 = 6$ Hz
$x_2(t) = \sin(2\pi f_2 t)$
$f_2 = 5$ Hz
$x_1(t) + x_2(t)$
$f_{leb} = f_1 - f_2$
T_{leb}
$x(t)$
t
6
5
4
3
2
1
0
-1
-2
0,0
0,5
1,0
1,5
2,0
2,5
3,0

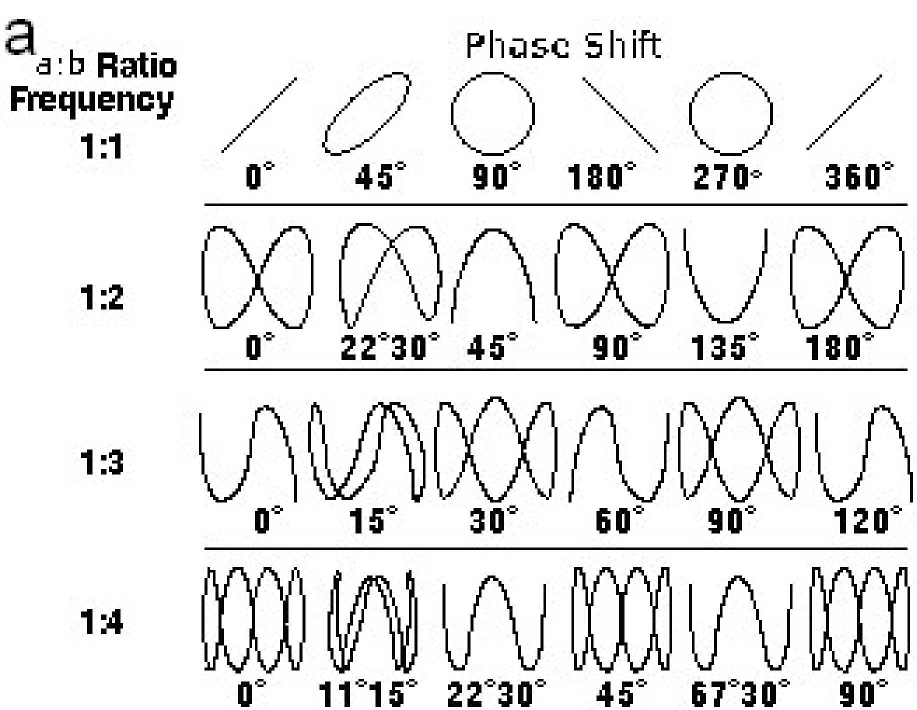
Superposition of perpendicular oscillations, the *Lissajous-curves*.

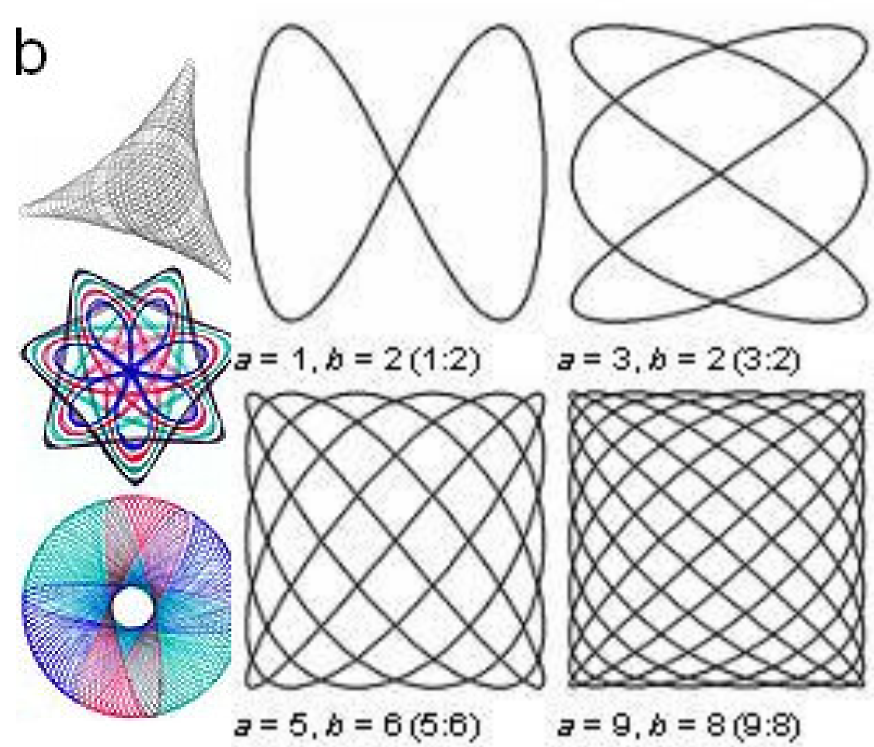
a) The frequencies are the same. The resultant oscillation is generally “elliptic oscillation” (elliptically polarized oscillation):

$$x = A\sin\omega t \qquad y = B\sin(\omega t + \alpha)$$

b) The frequencies are different.

$$x = A\sin\omega_a t \qquad y = B\sin(\omega_b t + \alpha)$$

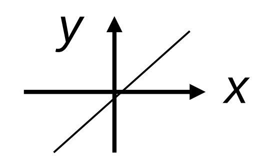
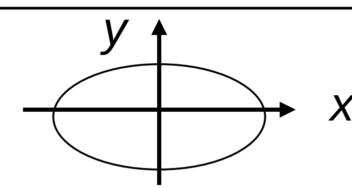
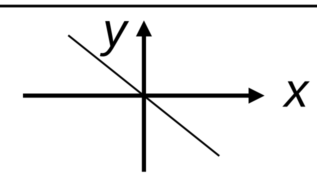
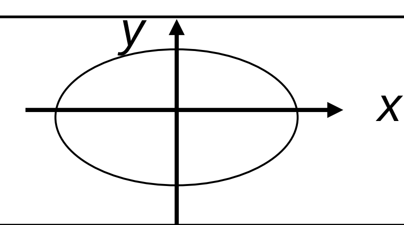




Example: elliptically polarized oscillation

$$x = A\sin\omega t \qquad y = B\sin(\omega t + \alpha)$$

$$\sin(\omega t + \alpha) = \sin\omega t \cdot \cos\alpha + \cos\omega t \cdot \sin\alpha$$

Initial phase α	x	y	Equation of the curve	Plot
0^{o}	$A\cdot\sin(\omega t)$	$B\cdot\sin(\omega t)$	$\frac{y}{x} = \frac{B}{A}$	
90^{o}	$A\cdot\sin(\omega t)$	$B\cdot\cos(\omega t)$	$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 = 1$	
180^{o}	$A\cdot\sin(\omega t)$	$-B\cdot\sin(\omega t)$	$\frac{y}{x} = -\frac{B}{A}$	
270^{o}	$A\cdot\sin(\omega t)$	$-B\cdot\cos(\omega t)$	$\left(\frac{x}{A}\right)^2 + \left(\frac{y}{-B}\right)^2 = 1$	

Example: elliptically polarized oscillation for arbitrary initial phase

$$x = A\sin\omega t \qquad y = B\sin(\omega t + \alpha) = B\cdot\sin\omega t\cdot\cos\alpha + B\cdot\cos\omega t\cdot\sin\alpha$$

By elimination of the time, *t* ((sin *ωt*) and és (cos *ωt*) are substituted by expressions of *x*):

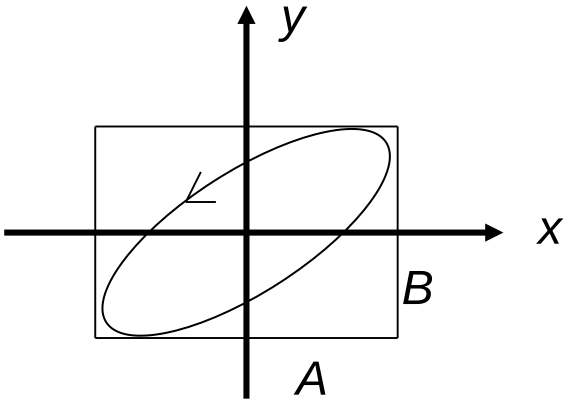
$$\frac{y}{B} = \frac{x}{A}\cos\alpha + \sqrt{1-\frac{x^2}{A^2}}\cdot\sin\alpha \longrightarrow \left(\frac{y}{B} - \frac{x}{A}\cdot\cos\alpha\right)^2 = \left(1-\frac{x^2}{A^2}\right)\cdot\sin^2\alpha$$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{2xy}{AB}\cos\alpha = \sin^2\alpha$$

This is the equation of a second order curve, which is an ellipse. The *x*,*y* coordinates can take finite (*A* or *B*) values only.

Two perpendicular and harmonic vibrations of equal amplitudes and frequencies result in elliptic oscillation.

The frame of the ellipse is a rectangle of 2*A* and 2*B* side lengths with its center in the origin.



Decomposition of the oscillations into harmonic oscillations; the Fourier theorem.

The Fourier-theorem: if $g(t)$ is a periodic function of the time, $g(t) = g(t+T)$ then it can be decomposed into the sum of sine and cosine functions in one way only where the amplitudes (A_i and B_i, i = 0,1,2,...) of the harmonics (components) are different and the frequencies $\omega_i = i \cdot \omega$ (i = 1,2,...) are integral multiples of the fundamental frequency ω:

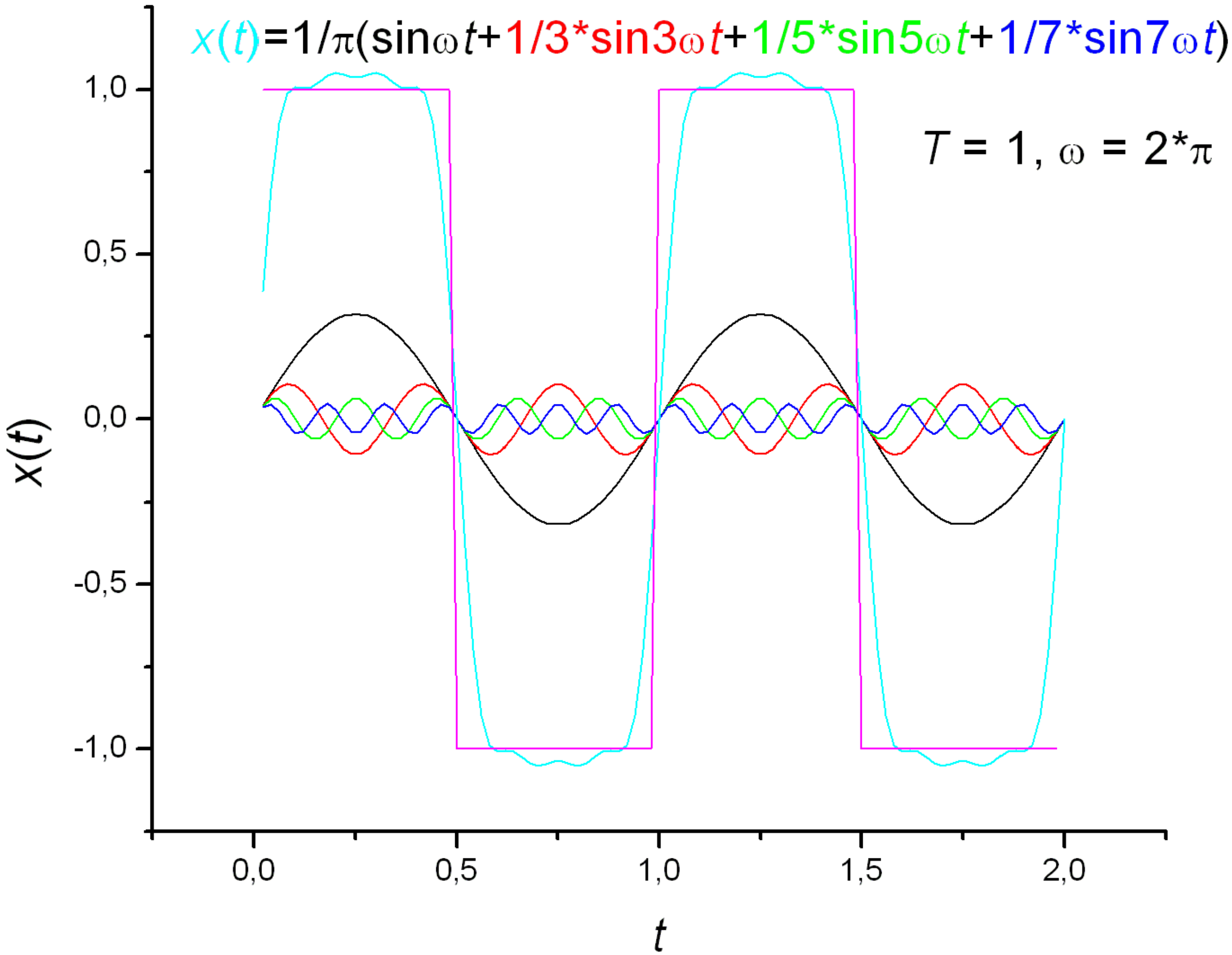
$$g(t) = A_0/2 + A_1\cos\omega t + A_2\cos 2\omega t + \ldots + B_1\sin\omega t + B_2\sin 2\omega t + \ldots$$

where the coefficients (amplitudes) *A*i and *B*i are determined by the following integrals:

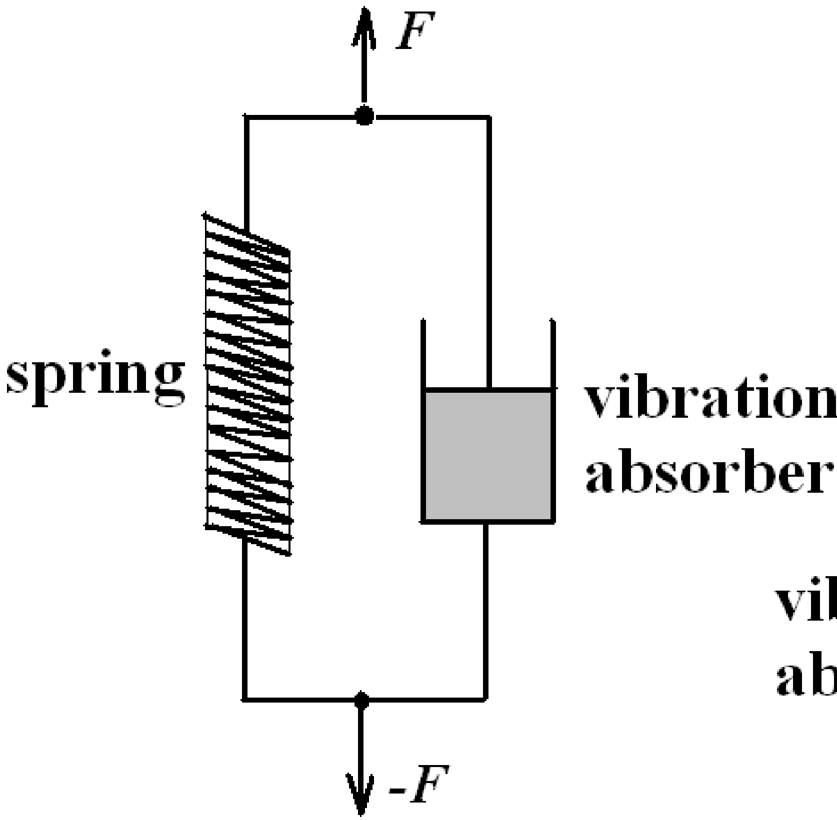
$$A_i = \frac{2}{T}\int_0^T g(t)\cos(i\omega t)\,dt \qquad B_i = \frac{2}{T}\int_0^T g(t)\sin(i\omega t)\,dt \qquad (i = 0,1,2,3,\ldots)$$

Example: Fourier-analísys of the step function

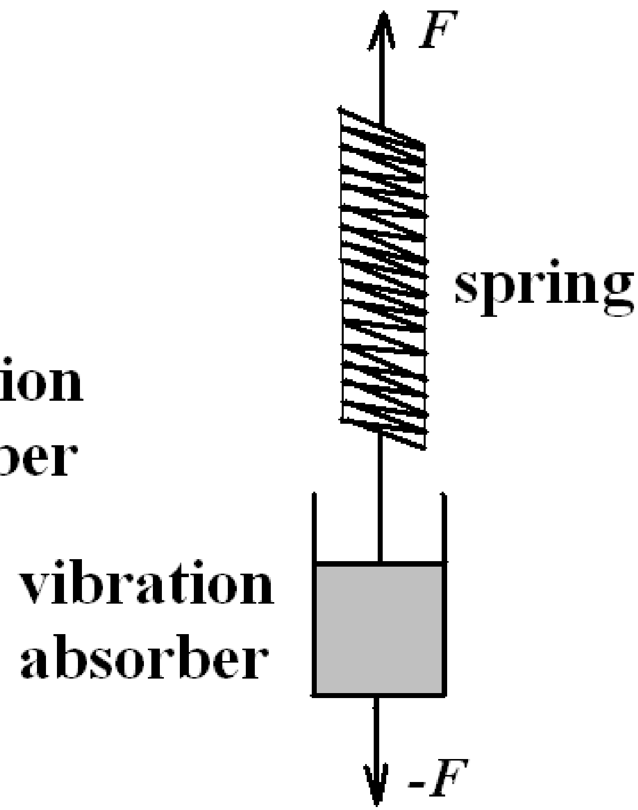
$$x(t) = \frac{1}{\pi}(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + \ldots)$$



Damped oscillations.



Voigt body



Maxwell body

Voigt body

force: $F = F_1 + F_2$

displacement: $x = x_1 = x_2$

Maxwell body

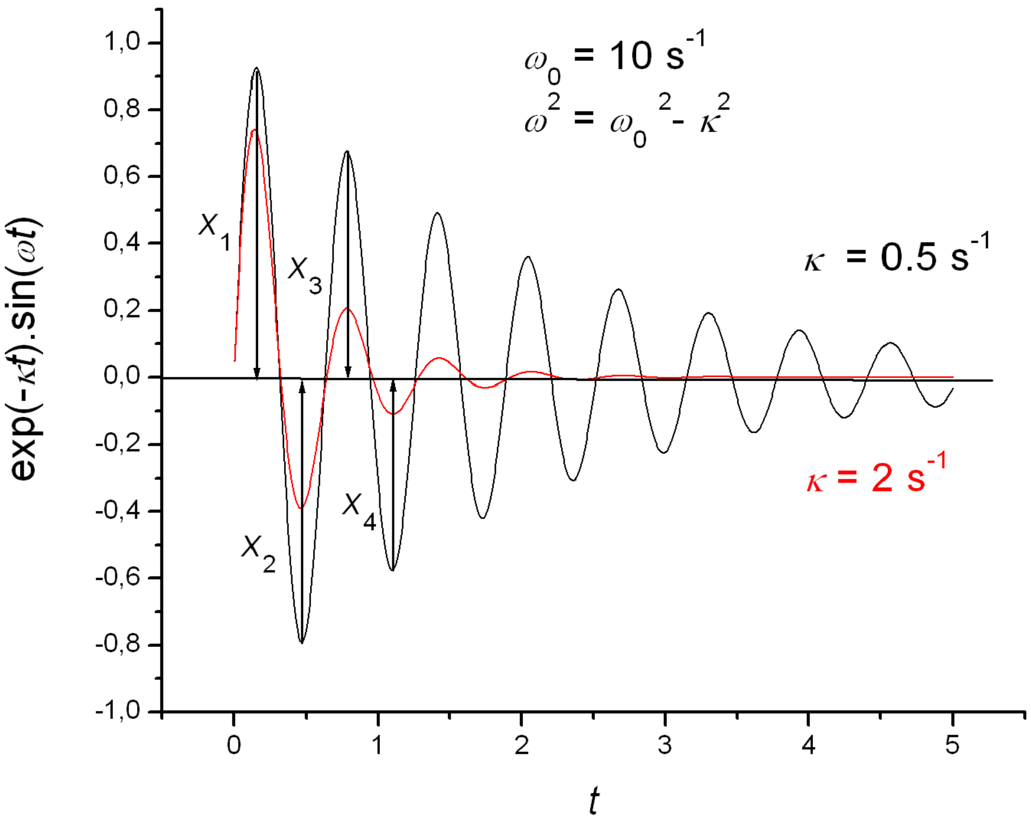
$F = F_1 = F_2$

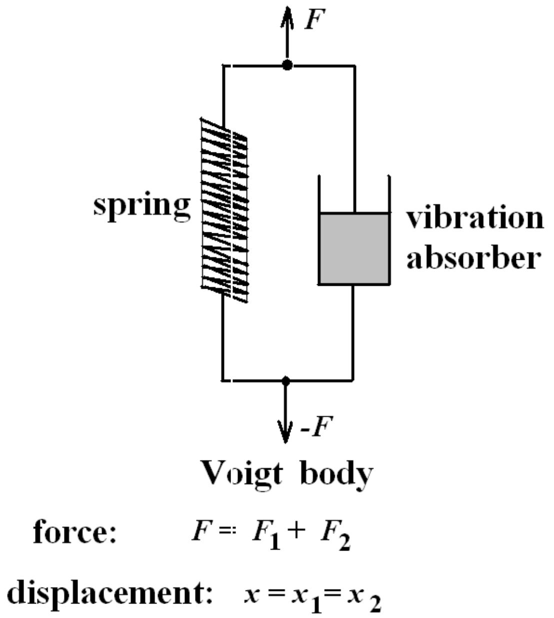
$x = x_1 + x_2$

Oscillation of Voigt-body

(the damping is small)

$$\left(\frac{\eta}{2m}=\right)\kappa<\omega_0\left(=\sqrt{\frac{k}{m}}\right)$$





$$m\cdot\frac{d^2x}{dt^2}=-k\cdot x-\eta\cdot\frac{dx}{dt}$$

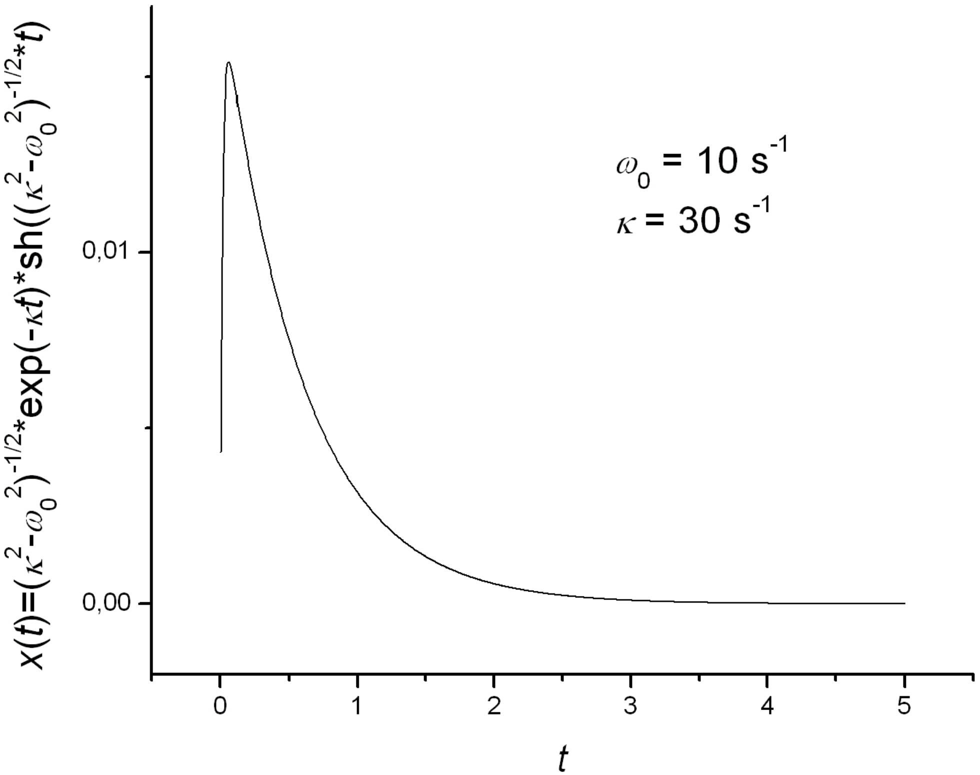
$$\omega=\sqrt{\omega_0^2-\kappa^2}$$

$$x=A\cdot e^{-\kappa t}\cdot\sin(\omega t+\alpha)$$

Aperiodic movement of the Voigt-body, the damping is large $\kappa > \omega_0$

The analytic solution with $x(t=0)=0$ and $v(t=0)=v_0$ initial conditions:

$$x = \frac{v_0}{\sqrt{\kappa^2 - \omega_0^2}} e^{-\kappa t} \mathrm{sh}\left(\sqrt{\kappa^2 - \omega_0^2}\, t\right)$$



The initially deviated body approaches the equilibrium position from one side only and does not swing over the other side at all (see e.g. the movement of a deviated pendulum moving in an extremely viscous fluid (e.g. In honey).

Forced (induced) vibrations.

If the system is exposed to periodic and external force (which will act not promptly but continuously), then the system will make forced (induced) vibrations. The dynamic equation of the motion can be created from that of the damped oscillation extended by the periodic external driving force:

$$m \cdot \frac{d^2x}{dt^2} = -k \cdot x - \eta \cdot \frac{dx}{dt} + F_0 \sin \omega t$$

In the case of $\kappa < \omega_0$ (the damping is small), the general solution can be given in analytical form:

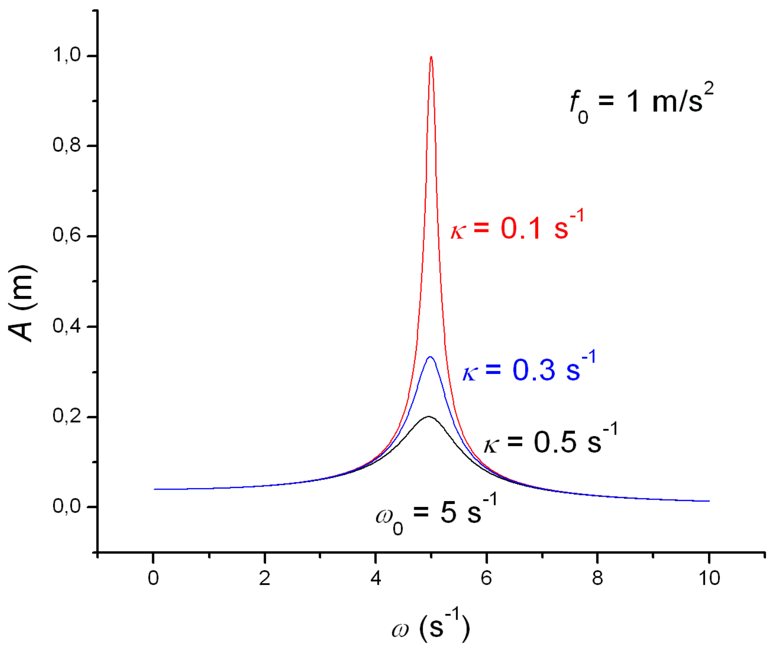
$$x(t) = A\cos(\omega t - \varphi) + ae^{-\kappa t} \sin\left(\sqrt{\omega_0^2 - \kappa^2} \cdot t + \alpha\right)$$

where

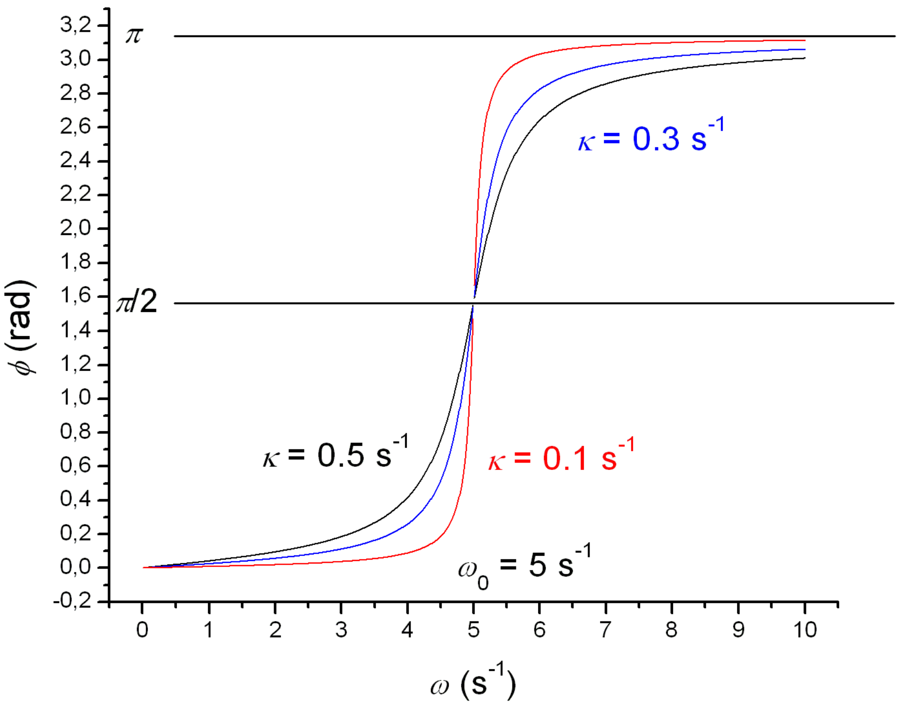
$$A = \frac{F_0 / m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\kappa^2\omega^2}} \qquad \mathrm{tg}\,\varphi = \frac{2\kappa\omega}{\omega_0^2 - \omega^2}$$

Resonance

Amplitude



Phase



$$A = \frac{F_0 / m}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\kappa^2\omega^2}}$$

$$\mathrm{tg}\,\varphi = \frac{2\kappa\omega}{\omega_0^2 - \omega^2}$$

Summary of (absolute) basic definitions and expressions

1. The *amplitude* of an oscillation (A) is the maximum displacement from the equilibrium position.

2. The *period* of oscillatory motion (T) is the shortest time that elapses between successive occurrences of the same configuration.

3. The *frequency* of oscillatory motion (f) is defined as the number of oscillations that occur per unit time (this is also known as the *linear frequency*, to distinguish it from the *angular frequency* ω): $f = 1/T$ (and $\omega = 2\pi/T$).

4. *Harmonic motion* is a special type of oscillatory motion that results from a restoring force, F (or torque) which is directly proportional to the displacement from equilibrium (Δx) and directed always towards the equilibrium position: $F = -k \cdot \Delta x$. The proportionality factor, k is called *directional force constant.* Simple harmonic motion is characterized by solutions that involve the „harmonic functions" (sine and cosine) and a period that is independent of the amplitude.

5. The *phase constant* (α) is a constant in the argument of the sine (or cosine) function used to describe the oscillatory motion: $x = A \sin(\omega \cdot t + \alpha)$. It is determined by the initial state of the system.

6. A *simple* (or *mathematical*) *pendulum* consists of a point particle of mass m, swinging from a massless string of length l. The period T of small oscillations is independent of the mass and the amplitude, and depends only on the length of the string and the acceleration of gravity, g: $T = 2\pi\sqrt{l/g}$.
If the rotational inertia of the pendulum differs from that of a point particle, we have a *physical pendulum*. If the object is pivoted at a distance L from its center of mass and allowed to swing with small amplitude, the period T of oscillations is $T = 2\pi\sqrt{I/mgL}$, where I is the moment of inertia for rotations about the pivot.

7. Oscillations with decreasing amplitude constitute *damped harmonic motion* during which dissipative forces (friction, fluid viscosity, etc.) decrease the amplitude of the oscillation with time because the mechanical energy of the system is gradually converted into thermal energy.

8. To maintain constant amplitude if damping forces are present, it is necessary to replenish the mechanical energy of the system. The resulting oscillations are known as *driven harmonic motion.*

9. If the frequency of the driving force matches the natural frequency of oscillation of the system, the system is in *resonance*. In resonance, the amplitude of the oscillations reaches a maximum and the phase shift between the oscillating driving force and the oscillating motion of the system is $\pi/2$ (= 90^o).

Problems for home works and/or seminars

1) A basketball player of mass 100 kg lifts his center of mass by 1 m on jumping up. On landing, he needs 10 cm (elastic landing) or 1 cm (inelastic damping) path to damp completely his speed. Estimate the forces of damping (evoked in the vertebral column) on landing!

2) The mechanical role of the tiny bones in the human middle ear is approximated by the following model: a point mass of m = 2 mg is anchored to the ear drum and to the oval window by two springs of directional forces $k1$ = 72 N/m and $k2$ = 7.2 N/m, respectively. How much is the (self) frequency of this system?

3) The mass of an unloaded car is 800 kg. The body of the car will sink 6 cm after getting in 5 persons of total mass of 500 kg. How much is the time of period of vibration of the unloaded car and loaded with passengers?

4) A log floating on the surface of the water is pressed slightly down and left alone. Determine the frequency of the swinging log!

5) The walking can be considered as movement of the unloaded leg as physical pendulum from the back ahead in a passive way (without intervenience of the muskels of this leg). Estimate the speed of walking if the length of one foot step is s = 0.8 m and the unloaded leg of length l = 1 m is swinging around the hip as pivot axis!

Problems for home works and/or seminars

6) Demonstrate the basic laws of superposition of vibrations using computer graphics methods! Construct simple Lissajous-curves!

7) The Fourier-decomposition of triangle-shaped vibration of unit amplitude is

$$x(t) = \frac{8}{\pi^2}\left(\sin\omega t - \frac{1}{3^2}\sin 3\omega t + \frac{1}{5^2}\sin 5\omega t - \frac{1}{7^2}\sin 7\omega t + \ldots\right)$$

By plotting the (sum of the) components, demonstrate that a few components give acceptable approximation.

8) Give the equation of motion of a point mass attached to a spring and a damper coupled in series (Maxwell body)!

9) A mosquito is hitting with his leg the Szeged downtown bridge with a frequency identical with the self frequency of the bridge. In contradiction to the expectations of the catastrophe of resonance, the bridge will not collapse. Why not?

10) The springs of the wagons absorb periodic shocks at the connections of the rails and vibrations will be evoked. The compression of the springs is 1.6 μm upon 1 N load, the mass of the wagon is 22 tons and the length of the rails is 18 m. At what speed is the amplitude of the vibration the largest?